Equivalences of linear functional systems

Thomas Cluzeau, Alban Quadrat

Equivalences of linear functional systems

T. Cluzeau, A. Quadrat

RESEARCH REPORT
N° RR-9000
December 2016
Project-Teams Non-A

ISSN 0249-6399   ISRN INRIA/RR--RR-9000--FR+ENG

Equivalences of linear functional systems

T. Cluzeau[*], A. Quadrat[†]

Project-Teams Non-A

Research Report n° RR-9000 — December 2016 — 26 pages

Abstract: Within the algebraic analysis approach to linear systems theory, we investigate the *equivalence problem* of linear functional systems, i.e., the problem of characterizing when all the solutions of two linear functional systems are in a one-to-one correspondence. To do that, we first provide a new characterization of isomorphic finitely presented modules in terms of inflations of their presentation matrices. We then prove several isomorphisms which are consequences of the *unimodular completion problem*. We then use these isomorphisms to complete and refine existing results concerning *Serre's reduction problem*. Finally, different consequences of these results are given. All the results obtained in this paper are algorithmic for rings for which Gröbner basis techniques exist and the computations can be performed by the Maple packages OreModules and OreMorphisms.

Key-words: Linear systems theory, equivalence problem, control theory, algebraic analysis, computer algebra

[*] Université de Limoges ; CNRS ; XLIM UMR 7252, 123 avenue Albert Thomas, 87060 Limoges Cedex, France. thomas.cluzeau@unilim.fr.

[†] Inria Lille - Nord Europe, Non-A project, Parc Scientifique de la Haute Borne, 40 Avenue Halley, Bat. A - Park Plaza, 59650 Villeneuve d'Ascq, France. Email: alban.quadrat@ inria.fr.

Equivalences des systèmes linéaires fonctionnels

Résumé : Dans l'approche de la théorie des systèmes linéaires par l'analyse algébrique, nous étudions *le problème d'équivalence* des systèmes linéaires fonctionnels, c'est-à-dire le problème de caractériser quand toutes les solutions de deux systèmes linéaires fonctionnels sont en bijection. Pour cela, nous donnons tout d'abord une nouvelle caractérisation de l'isomorphisme entre deux modules de présentation finie en terme d'extensions de leurs matrices de présentation. Nous prouvons ensuite différents isomorphismes qui sont des conséquences du *problème de complétion unimodulaire*. Nous les utilisons alors pour compléter et raffiner des résultats existants sur le *problème de la réduction de Serre*. Finalement, différentes conséquences de ces résultats sont données. Tous les résultats obtenus dans ce papier sont algorithmiques pour des anneaux sur lesquels les techniques de bases de Gröbner existent et les calculs peuvent être obtenus par les packages Maple OREMODULES et OREMORPHISMS.

Mots-clés : Théorie des systèmes linéaires, problème d'équivalence, théorie du contrôle, analyse algébrique, calcul formel

1 Introduction

Mathematical systems which are studied in control theory, mathematical physics, and engineering sciences can usually be modeled by systems of *functional equations*, namely, equations whose unknowns are functions. These functions can depend on one or more continuous or discrete variables. Standard examples of functional equations are ordinary differential (OD) or partial differential (PD) equations, (partial) difference equations, differential time-delay equations, ... Functional systems can be studied by a large amount of mathematical theories as functional analysis, numerical analysis, differential geometry, ... In this paper, we focus on linear functional systems, i.e., on the case where the functional equations are linear. In particular, we use the *algebraic analysis approach* to linear systems theory to study built-in properties of linear functional systems. Algebraic analysis has been developed by Malgrange, Bernstein, Sato, Kashiwara, ... For more details, see [14, 15, 17, 19, 21] and the references therein.

We shall study here linear functional systems which can be written as $R\,\eta = 0$, where R is a $q \times p$ matrix with entries in a (noncommutative) polynomial ring D of functional operators (e.g., OD or PD operators, shift operators, difference operators, OD time-delay operators) and η is a vector of unknown functions which belong to a functional space (e.g., smooth functions, distributions, hyperfunctions). More precisely, if $\mathcal{F}$ is a left D-module (see, e.g., [16, 24]), then we can consider the following *linear system*

$$\ker_{\mathcal{F}}(R.) := \{\eta \in \mathcal{F}^p \mid R\,\eta = 0\},$$

also called a *behavior* in control theory (see [19] and the references therein). The algebraic analysis approach to linear systems theory (see [3, 13, 19, 21, 23] and the references therein) is based on the fact that the linear system $\ker_{\mathcal{F}}(R.)$ can be studied by means of the *factor left D-module $M := D^{1\times p}/(D^{1\times q}\,R)$ finitely presented by the matrix R*. Indeed, Malgrange's isomorphism [17] states that we have

$$\ker_{\mathcal{F}}(R.) \cong \hom_D(M, \mathcal{F}),$$

where $\hom_D(M, \mathcal{F})$ denotes the abelian group (i.e., $\mathbb{Z}$-module) of all the left D-homomorphisms (i.e., left D-linear maps) from M to $\mathcal{F}$ (see Section 2 for more details). Hence, module properties of M and $\mathcal{F}$ are connected to system-theoretical properties of $\ker_{\mathcal{F}}(R.)$. Using constructive methods of *homological algebra* [24] for *Gröbner rings D* (namely, (noncommutative) polynomial rings for which *Gröbner bases* can be computed for all *admissible term orders* by means of *Buchberger's algorithm* [5]) [3, 6, 21], we can effectively characterize module properties of M which are important in control theory (see [3, 13, 19, 21, 23] and references therein). The corresponding algorithms are implemented in dedicated packages of computer algebra systems (e.g., OREMODULES [4] and OREMORPHISMS [7] developed in `Maple`, and OREALGEBRAICANALYSIS [12] developed in `Mathematica`).

The purpose of the paper is to use the algebraic analysis framework to consider the following three important issues in mathematical systems (resp., module) theory:

1. *Equivalence problem*: Recognize whether or not two linear systems (resp., modules) are isomorphic.

2. *Unimodular completion problem*: Inflate (if possible) a given (rectangular) matrix into a unimodular, namely, an invertible, (square) matrix.

3. *Serre's reduction problem*: Find an equivalent system defined by fewer equations and fewer unknowns.

The first contribution of the paper (see Theorem 1) provides an explicit characterization of isomorphic finitely presented modules in terms of inflations of their presentation matrices. This characterization yields a general characterization of equivalent linear systems which do not necessarily have the same number of unknowns and equations. A constructive version of the classical *Schanuel's lemma* (see, e.g., [24]) on the *syzygy modules* of these modules can then be found again as a direct application of Theorem 1. If D is a *stably finite ring* (e.g., a *noetherian ring*) (see, e.g., [16]) and one of the presentation matrices has full row rank, then this result yields a characterization of isomorphic modules in terms of the unimodular completion problem (which also characterizes Serre's reduction problem [1]). The second contribution (see Theorem 2) is to show how the completion problem induces isomorphisms between the different modules finitely presented by the matrices appearing in the inflations. This result can be seen as an extension of a result obtained for Serre's reduction problem in [1] (extension for non necessarily full row rank matrices). The results are illustrated by explicit examples where all the computations can be performed using the packages OREMODULES [4] and OREMORPHISMS [7].

The paper is organized as follows. In Section 2, we briefly review the algebraic analysis approach to linear systems theory. In Section 3, we recall useful results of [6] on homomorphisms and isomorphisms of finitely presented left D-modules. In Section 4, we give an explicit characterization of the inverse of an isomorphism and a characterization of isomorphic finitely presented modules in terms of inflations of their presentation matrices. Interesting consequences of this result in linear systems theory are then given. In Section 5, we give our second main result on the different isomorphisms induced by a solution to the unimodular completion problem. Finally, this result is applied to Serre's reduction problem to refine a result obtained in [1].

2 Linear functional systems and finitely presented left modules

In this section, we show how a linear system defines a *finitely presented left D-module* and conversely. This correspondence plays a fundamental role in what follows as linear systems will be studied by means of the corresponding modules.

Let D be a noetherian ring and $R \in D^{q\times p}$ a matrix defining the linear system

$$\ker_{\mathcal{F}}(R.) := \{\eta \in \mathcal{F}^p \mid R\,\eta = 0\}$$

for a certain left D-module $\mathcal{F}$ (see Section 1). Using the matrix $R \in D^{q\times p}$, we can define the following multiplication map:

$$\begin{array}{rccc} .R\colon & D^{1\times q} & \longrightarrow & D^{1\times p} \\ & \lambda & \longmapsto & \lambda\, R. \end{array}$$

Since D is a (noncommutative) ring and not a (skew) field, $D^{1\times q}$ and $D^{1\times p}$ are *(left/right) D-modules*. We recall that a module is an algebraic structure defined by the same conditions as those for a vector space but where the scalars belong to a ring and not a (skew) field (see, e.g., [16,24]). If M_1 and M_2 are two left D-modules, then a homomorphism f from M_1 to M_2, which is denoted by $f \in \hom_D(M_1, M_2)$, is a map $f : M_1 \longrightarrow M_2$ satisfying the following condition:

$$\forall\, d_1,\, d_2 \in D,\ \forall\, m_1,\, m_2 \in M_1 :\ f(d_1\, m_1 + d_2\, m_2) = d_1\, f(m_1) + d_2\, f(m_2).$$

For all $\lambda_1,\, \lambda_2 \in D^{1\times q}$ and for all $d_1,\, d_2 \in D$, we have

$$(.R)(d_1\, \lambda_1 + d_2\, \lambda_2) = d_1\, (\lambda_1\, R) + d_2\, (\lambda_2\, R) = d_1\, ((.R_1)(\lambda_1)) + d_2\, ((.R_2)(\lambda_2)),$$

i.e., $.R \in \hom_D(D^{1\times q}, D^{1\times p})$. Similarly, we can define homomorphisms for right D-modules. The image $\operatorname{im}_D(.R) := \{\mu\, R \mid \mu \in D^{1\times q}\}$ of $.R$, also simply denoted by $D^{1\times q}\, R$, is the left D-module formed by all the left D-linear combinations of the rows of the matrix R. The *cokernel* of $.R$ is defined by the following *factor* left D-module:

$$M := D^{1\times p}/(D^{1\times q}\, R).$$

Two vectors $\lambda_1,\, \lambda_2 \in D^{1\times p}$ are said to belong to the same *residue class*, which is denoted by $\pi(\lambda_1) = \pi(\lambda_2)$, if we have $\lambda_1 - \lambda_2 \in D^{1\times q}\, R$, i.e., if there exists $\mu \in D^{1\times q}$ such that $\lambda_1 = \lambda_2 + \mu\, R$. The left D-module M is then defined by all the $\pi(\lambda)$'s for $\lambda \in D^{1\times p}$ with the following two binary operations:

$$\forall\, \lambda_1,\, \lambda_2 \in D^{1\times p},\ d \in D :\ \pi(\lambda_1 + \lambda_2) := \pi(\lambda_1) + \pi(\lambda_2),\ \pi(d\, \lambda_1) := d\, \pi(\lambda_1).$$

We can check that $\pi(\lambda_1) + \pi(\lambda_2)$ and $d\, \pi(\lambda_1)$ do not depend on the choice of the *representatives* $\lambda_1,\, \lambda_2$ of the residues classes $\pi(\lambda_1)$ and $\pi(\lambda_2)$, which shows that the two above binary operations are well-defined on M and $\pi \in \hom_D(D^{1\times p}, M)$ is called the canonical projection onto M.

The left D-module M is said to be *finitely presented* and R is called a *presentation matrix* [16, 24]. Let us explicitly describe M by means of *generators* and *relations*. If $\{f_j\}_{j=1,\dots,p}$ denotes the *standard basis* of $D^{1\times p}$, namely, f_j is the row vector of length p formed by 1 at the j^{th} position and 0 elsewhere, and $y_j := \pi(f_j)$ for $j = 1, \dots, p$, then we claim that $\{y_j\}_{j=1,\dots,p}$ is a generator set for M. Indeed, an element $m \in M$ is of the form $m = \pi(\lambda)$ for a certain $\lambda := (\lambda_1 \ \dots \ \lambda_p) = \sum_{j=1}^{p} \lambda_j\, f_j \in D^{1\times p}$, which yields $m = \sum_{j=1}^{p} \lambda_j\, y_j$ since $\pi \in \hom_D(D^{1\times p}, M)$. A left/right D-module which admits a finite set of generators is said to be *finitely* *generated*. The y_j's are not left D-linearly independent since, if $R_{i\bullet}$ denotes the i^{th} row of R, using the fact that $R_{i\bullet} \in D^{1\times q}\, R$ and $\pi \in \hom_D(D^{1\times p}, M)$, we then obtain:

$$\forall\, i = 1, \ldots, q, \quad \sum_{j=1}^{p} R_{ij}\, y_j = \sum_{j=1}^{p} R_{ij}\, \pi(f_j) = \pi(R_{i\bullet}) = 0. \tag{1}$$

Hence, the set of generators $\{y_j\}_{j=1,\ldots,p}$ satisfies the *left D-linear relations* (1). If we note $y := (y_1 \ \ldots \ y_p) \in M^p$, then (1) can be rewritten as $R\, y = 0$.

If $\mathcal{F}$ is a left D-module, then we can define the following *behavior*

$$\ker_{\mathcal{F}}(R.) := \{\eta \in \mathcal{F}^p \mid R\,\eta = 0\},$$

i.e., the space of $\mathcal{F}$-solutions of $R\,\eta = 0$. We claim that there is an *isomorphism* (namely, an injective and a surjective homomorphism) between $\ker_{\mathcal{F}}(R.)$ and $\hom_D(M, \mathcal{F})$, which is denoted by $\ker_{\mathcal{F}}(R.) \cong \hom_D(M, \mathcal{F})$. Let us describe this isomorphism. If $\phi \in \hom_D(M, \mathcal{F})$, $\{y_j\}_{j=1,\ldots,p}$ the set of generators of M defined above, and $\eta_j := \phi(y_j)$ for $j = 1, \ldots, p$, then, using (1), we get

$$\forall\, i = 1, \ldots, q, \quad \sum_{j=1}^{p} R_{ij}\, \eta_j = \sum_{j=1}^{p} R_{ij}\, \phi(y_j) = \phi\left(\sum_{j=1}^{p} R_{ij}\, y_j\right) = \phi(0) = 0,$$

i.e., $\eta := (\eta_1 \ \ldots \ \eta_p)^T \in \ker_{\mathcal{F}}(R.)$. Conversely, if $\eta = (\eta_1 \ \ldots \ \eta_p)^T \in \ker_{\mathcal{F}}(R.)$, then we can define $\phi_\eta : M \longrightarrow \mathcal{F}$ by $\phi_\eta(\pi(\lambda)) := \lambda\,\eta$ for all $\lambda \in D^{1\times p}$. If $\pi(\lambda) = \pi(\lambda')$, then there exists $\mu \in D^{1\times q}$ such that $\lambda = \lambda' + \mu\, R$, which yields $\lambda\,\eta = \lambda'\,\eta$ since $R\,\eta = 0$, which shows that $\phi_\eta(\pi(\lambda)) = \phi_\eta(\pi(\lambda'))$, i.e., ϕ_η does not depend on the representative λ of $\pi(\lambda)$. Clearly, we have $\phi_\eta \in \hom_D(M, \mathcal{F})$. Now, if $\eta \in \ker_{\mathcal{F}}(R.)$, then we get $\phi_\eta(y_j) = \phi_\eta(\pi(f_j)) = f_j\,\eta = \eta_j$, which shows that the additive map

$$\begin{array}{rcl} \chi : \ker_{\mathcal{F}}(R.) & \longmapsto & \hom_D(M, \mathcal{F}) \\ \eta & \longmapsto & \phi_\eta, \end{array} \tag{2}$$

is injective. It is also surjective since, for every $\phi \in \hom_D(M, \mathcal{F})$, we can define $\eta := (\phi(y_1) \ \ldots \ \phi(y_p))^T \in \ker_{\mathcal{F}}(R.)$ and we have

$$\forall\, \lambda \in D^{1\times p}, \quad \phi_\eta(\pi(\lambda)) := \lambda\,\eta = \sum_{j=1}^{p} \lambda_j\, \eta_j = \phi\left(\sum_{j=1}^{p} \lambda_j\, y_j\right) = \phi(\pi(\lambda)),$$

which shows that $\phi = \phi_\eta = \chi(\eta)$ and finally proves that we have the isomorphism:

$$\ker_{\mathcal{F}}(R.) \cong \hom_D(M, \mathcal{F}).$$

Remark 1. We note that $\phi_{d\,\eta}(\pi(\lambda)) = \lambda\, d\,\eta$ is usually different from $d\,\lambda\,\eta = d\,\phi_\eta(\pi(\lambda))$ when D is a noncommutative ring, i.e., χ is not a left D-homomorphism. It is only an abelian group (i.e., a $\mathbb{Z}$-module) homomorphism between abelian groups (i.e., $\mathbb{Z}$-modules). If D is a k-algebra, where k is a field, then $\hom_D(M, \mathcal{F})$ inherits a k-vector space structure and χ is then an isomorphism of k-vector spaces.

Hence, the behavior $\ker_{\mathcal{F}}(R.)$ is the "dual" of the finitely presented left D-module $M := D^{1\times p}/(D^{1\times q}\, R)$ [14, 15]. We pass from a finitely presented left D-module M (the algebraic side of a linear system) to a behavior $\ker_{\mathcal{F}}(R.)$ (the analytical side of a linear system) by applying the *contravariant functor* $\hom_D(\,\cdot\,, \mathcal{F})$ (see, e.g., [24]). In particular, the algebraic study of M yields information on the behavior $\ker_{\mathcal{F}}(R.)$. For more details, see [3, 6, 13, 19, 21] and the references therein.

The algebraic analysis approach to linear systems theory is very general. In mathematical systems theory and control theory, we usually focus on particular classes of linear functional systems such as linear OD systems or DTD systems. In this case, we consider an algebra D of functional operators such as *skew polynomial rings*, *Ore algebras*, *Ore extensions*, ... For more details, see [3, 5, 12, 18] and the references therein. Let us give an explicit example.

Example 1. Let us consider the following linear DTD system

$$\begin{cases} \dot{x}_1(t) = x_2(t) + u(t), \\ \dot{x}_2(t) = x_1(t - 3\,h) + x_1(t - 2\,h) + u(t), \end{cases} \tag{3}$$

where h is a non-negative real, i.e., $h \in \mathbb{R}_{\geq 0}$. Let us consider the differential operator $\partial\, z(t) := \dot{z}(t)$ and the time-delay operator $\delta\, z(t) := z(t - h)$ which satisfy

$$(\partial\, \delta)\, z(t) = \partial\, z(t - h) = \dot{z}(t - h) = (\delta\, \partial)\, z(t),$$

i.e., on the level of operators, we have $\partial\, \delta = \delta\, \partial$, where the product stands for the composition of operators. Let $D := \mathbb{Q}[\partial, \delta]$ be the commutative polynomial algebra formed by the operators in ∂ and δ with coefficients in $\mathbb{Q}$. An element $d \in D$ is of the form $d = \sum_{0 \leq i,j \leq r} a_{ij}\, \partial^i\, \delta^j$, where $r \in \mathbb{N}$, $a_{ij} \in \mathbb{Q}$, and $\partial^i\, z(t) = z^{(i)}(t)$ (resp., $\delta^i\, z(t) = z(t - i\,h)$) is the i^{th} composition of ∂ (resp., of δ). Then, (3) can be rewritten as $R\,\eta = 0$, where $\eta := (x_1 \quad x_2 \quad u)^T$ and:

$$R := \begin{pmatrix} \partial & -1 & -1 \\ -\delta^2\,(\delta + 1) & \partial & -1 \end{pmatrix} \in D^{2 \times 3}.$$

We consider the finitely presented D-module $M := D^{1\times 3}/(D^{1\times 2}\, R)$, $\{f_j\}_{j=1,2,3}$ is the standard basis of $D^{1\times 3}$, $x_1 := \pi(f_1)$, $x_2 := \pi(f_2)$, and $u := \pi(f_3)$, where $\pi : D^{1\times 3} \longrightarrow M$ is the canonical projection. Then, as previously shown, $\{x_1,\, x_2,\, u\}$ is a set of generators of M which satisfies the following D-linear relations:

$$\begin{cases} \partial x_1 - x_2 - u = 0, \\ \partial x_2 - \delta^2\,(\delta + 1)\, x_1 - u = 0. \end{cases}$$

It is important to note that x_1, x_2, and u are not functions but only the "abstract" generators of M. To get functions, i.e., elements of a functional space $\mathcal{F}$ having a D-module structure (e.g., $\mathcal{F} := C^{\infty}(\mathbb{R})$), we have to consider $\hom_D(M, \mathcal{F}) \cong \ker_{\mathcal{F}}(R.) = \{\eta = (x_1\ x_2\ u)^T \in \mathcal{F}^3 \mid R\,\eta = 0\}$. Dualizing M with coefficients in $\mathcal{F}$, the generators of M are then mapped to functions, i.e., $x_1 \longmapsto x_1(\cdot) \in \mathcal{F}$, $x_2 \longmapsto x_2(\cdot) \in \mathcal{F}$ and $u \longmapsto u(\cdot) \in \mathcal{F}$, satisfying (3).

For more examples, see [3, 6, 21] and the references therein.

Finally, let us shortly introduce a few basic concepts of *homological algebra* (see, e.g., [24]) which will be used thereafter. A sequence of left/right D-modules $\{M_i\}_{i \in \mathbb{Z}}$ and of left/right D-homomorphisms $\{f_i \in \hom_D(M_i, M_{i-1})\}_{i \in \mathbb{Z}}$ are called a *complex* of left/right D-modules if we have $f_i \circ f_{i+1} = 0$ for all $i \in \mathbb{Z}$, i.e., if we have $\operatorname{im} f_{i+1} \subseteq \ker f_i$ for $i \in \mathbb{Z}$. The complex is then denoted by:

$$\ldots \xrightarrow{f_{i+2}} M_{i+1} \xrightarrow{f_{i+1}} M_i \xrightarrow{f_i} M_{i-1} \xrightarrow{f_{i-1}} \ldots$$

The above complex is said to be an *exact sequence* if $\ker f_i = \operatorname{im} f_{i+1}$ for all $i \in \mathbb{Z}$. For instance, using the fact that $\operatorname{coker}_D(.R) := D^{1\times p}/\operatorname{im}_D(.R) = M$, we get the following exact sequence

$$0 \longrightarrow \ker_D(.R) \xrightarrow{i} D^{1\times q} \xrightarrow{.R} D^{1\times p} \xrightarrow{\pi} M \longrightarrow 0,$$

where i is the standard injection and $\ker_D(.R) := \{\mu \in D^{1\times q} \mid \mu\, R = 0\}$ is the left D-module, called the *second syzygy module* of M, generated by all the D-linear combinations among the rows of R. If the rows of R are D-linearly independent, i.e., $\ker_D(.R) = 0$, then we say that R has *full row rank*.

An exact sequence of the form $0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$, i.e., where g is surjective ($\operatorname{im} g = \ker 0 = M''$), $\ker g = \operatorname{im} f$, and f injective ($\ker f = \operatorname{im} 0 = 0$), is called a *short exact sequence*. For instance, if R has full row rank, then we have the following short exact sequence of left D-modules:

$$0 \longrightarrow D^{1\times q} \xrightarrow{.R} D^{1\times p} \xrightarrow{\pi} M \longrightarrow 0. \tag{4}$$

Example 2. We consider again Example 1. Let us check that R has full row rank. We have $\mu := (\mu_1 \quad \mu_2) \in \ker_D(.R)$ if and only if

$$\left\{ \begin{array}{l} \mu_1\,\partial - \mu_2\,\delta^2\,(\delta+1) = 0, \\ -\mu_1 + \mu_2\,\partial = 0, \\ \mu_1 + \mu_2 = 0, \end{array} \right. \quad \Longrightarrow \quad \left\{ \begin{array}{l} \mu_1 = -\mu_2, \\ \mu_2\,(\partial+1) = 0, \end{array} \right.$$

which yields $\mu_2 = 0$ since $D := \mathbb{Q}[\partial, \delta]$ is an integral domain (i.e., D does not contain nonzero zero-divisors), and thus we get $\mu = 0$. Hence, we have the short exact sequence (4) with $p = 3$ and $q = 2$.

If D is a Gröbner ring, then *elimination techniques* (e.g., *Gröbner bases*, *Janet bases*, ...) can be used to compute $\ker_D(.R)$ (see [3,21] and the references therein). Indeed, a set of generators of $\ker_D(.R)$ corresponds to a set of generators of the *compatibility conditions* $\mu\,\zeta = 0$ of the inhomogeneous linear system $R\,\eta = \zeta$. Thus, we have to eliminate η from $R\,\eta = \zeta$ to get a set of generators for $\ker_D(.R)$. For more details, see, e.g., [3,21] and the OREMODULES package [4].

Example 3. We consider again (3) with the output $y(t) := x_1(t) + x_2(t)$, i.e.:

$$\left(\begin{array}{cc} \partial & -1 \\ -\delta^2\,(\delta+1) & \partial \\ 1 & 1 \end{array} \right) \left(\begin{array}{c} x_1(t) \\ x_2(t) \end{array} \right) = \left(\begin{array}{c} u(t) \\ u(t) \\ y(t) \end{array} \right).$$

To simplify (for instance, for an observability test), let us suppose that we have $u = 0$ and $y = 0$ so that we get the following linear DTD system:

$$\left(\begin{array}{cc} \partial & -1 \\ -\delta^2\,(\delta+1) & \partial \\ 1 & 1 \end{array} \right) \left(\begin{array}{c} x_1(t) \\ x_2(t) \end{array} \right) = 0. \tag{5}$$

Let $R \in D^{3\times 2}$ be the above matrix of DTD operators and $M := D^{1\times 2}/(D^{1\times 3}\,R)$ the D-module associated with (5). Using elimination techniques (see, e.g., [3,21]) and their implementations in the OREMODULES package [4], we can check that we have $\ker_D(.R) = \operatorname{im}_D(.R_2)$, where:

$$R_2 := (\partial + \delta^2\,(\delta+1) \quad \partial + 1 \quad -\partial^2 + \delta^2\,(\delta+1)) \in D^{1\times 3}.$$

The row vector R_2 generates the D-module $\ker_D(.R)$ formed by the D-linear relations among the rows of R. We can check again that $R_2\,\zeta = 0$ generates the compatibility conditions of $R\,\eta = \zeta$. We note that $.R_2 \in \hom_D(D, D^{1\times 3})$ is injective since $\nu\,R_2 = 0$ yields $\nu\,(\partial+1) = 0$, and thus we get $\nu = 0$ since D is an integral domain. Then, we obtain the following long exact sequence of D-modules

$$0 \longrightarrow D \xrightarrow{.R_2} D^{1\times 3} \xrightarrow{.R} D^{1\times 2} \xrightarrow{\pi} M \longrightarrow 0,$$

called a *finite free resolution* of the D-module M (see, e.g., [3,21,24]).

In Example 3, the D-module $\ker_D(.R)$ is a finitely generated D-module because $\ker_D(.R)$ is a D-submodule of the *noetherian* D-module $D^{1\times 3}$ (which is a direct sum of the *noetherian ring* $D := \mathbb{Q}[\partial, \delta]$). For more details, see, e.g., [16,24]. In what follows, we shall assume that D is a *noetherian ring*, namely, every left/right ideal of D is finitely generated as a left/right D-module (see, e.g., [16,24]). Then, for every matrix $R \in D^{q\times p}$, there always exists $R_2 \in D^{r\times q}$ (possibly reduced to 0) such that $\ker_D(.R) = \operatorname{im}_D(.R_2)$.

3 Homomorphisms of behaviors/finitely presented left modules

Let $R \in D^{q\times p}$ and $R' \in D^{q'\times p'}$ be two matrices respectively defining the linear systems $R\,\eta = 0$ and $R'\,\eta' = 0$. In this section, we review results on transformations which map the $\mathcal{F}$-solutions of the first system to $\mathcal{F}$-solutions of the second one.

As learned in Section 2, we can define the two finitely presented left D-modules $M := D^{1\times p}/(D^{1\times q}\, R)$ and $M' := D^{1\times p'}/(D^{1\times q'}\, R')$ which are associated with the above linear systems. Now, composing $\phi' \in \hom_D(M', \mathcal{F}) \cong \ker_{\mathcal{F}}(R'.)$ with $f \in \hom_D(M, M')$, we obtain the following *commutative diagram*

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[dr, "\phi'\circ f"'] & M' \arrow[d, "\phi'"] \\ & \mathcal{F} \end{tikzcd}$$

and we get $f^\star(\phi') := \phi' \circ f \in \hom_D(M, \mathcal{F}) \cong \ker_{\mathcal{F}}(R.)$. If $\{y_j := \pi(f_j)\}_{j=1,\ldots,p}$ (resp., $\{y'_k := \pi(f'_k)\}_{k=1,\ldots,p'}$) is the set of generators of M (resp., M') defined as in Section 2, a solution $\eta' := (\phi'(y'_1) \;\ldots\; \phi'(y'_{p'}))^T$ of $R'\,\eta' = 0$ is sent to the solution $\eta := (\phi'(f(y_1)) \;\ldots\; \phi'(f(y_p)))^T$ of $R\,\eta = 0$. To get an explicit description of η in terms of η', we have to explicitly know $f \in \hom_D(M, M')$, i.e., how f sends the y_j's to the y'_k's, i.e., to know the elements P_{jk} of D such that:

$$\forall\, j = 1, \ldots, p, \quad f(y_j) = \sum_{k=1}^{p'} P_{jk}\, y'_k. \tag{6}$$

Since f is a homomorphism, we have $f(0) = 0$. Using (1), for $i = 1, \ldots, q$, we get:

$$\begin{aligned} f\left(\sum_{j=1}^{p} R_{ij}\, y_j\right) = \sum_{j=1}^{p} R_{ij}\, f(y_j) &= \sum_{j=1}^{p} R_{ij}\left(\sum_{k=1}^{p'} P_{jk}\, y'_k\right) \\ &= \sum_{k=1}^{p'}\left(\sum_{j=1}^{p} R_{ij} P_{jk}\right) y'_k = 0. \end{aligned}$$

Using the fact that $y'_k := \pi'(f'_k)$, where $\pi' : D^{1\times p'} \longrightarrow M'$ is the canonical projection and $\{f'_k\}_{k=1,\ldots,p'}$ is the standard basis of $D^{1\times p'}$, we get

$$\pi'\left(\left(\sum_{j=1}^{p} R_{ij}\, P_{j1} \;\ldots\; \sum_{j=1}^{p} R_{ij}\, P_{jp'}\right)\right) = \sum_{k=1}^{p'}\left(\sum_{j=1}^{p} R_{ij} P_{jk}\right) y'_k = 0,$$

which shows the existence of row vectors $Q_i \in D^{1\times q'}$, $i = 1, \ldots, q$, such that:

$$\forall\, i = 1, \ldots, q, \quad \left(\sum_{j=1}^{p} R_{ij}\, P_{j1} \;\ldots\; \sum_{j=1}^{p} R_{ij}\, P_{jp'}\right) = Q_i\, R'.$$

If we note $P := (P_{jk})_{1\le j\le p, 1\le k\le p'} \in D^{p\times p'}$ and $Q := (Q_1^T \;\ldots\; Q_q^T)^T \in D^{q\times q'}$, then we obtain the following identity:

$$R\,P = Q\,R'. \tag{7}$$

Hence, we get that $f \in \hom_D(M, M')$ is defined by (6) where the P_{jk}'s satisfy (7).

Lemma 1 ([6]). *Let $M := D^{1\times p}/(D^{1\times q}\, R)$ (resp., $M' := D^{1\times p'}/(D^{1\times q'}\, R')$) be the left D-module finitely presented by $R \in D^{q\times p}$ (resp., $R' \in D^{q'\times p'}$) and $\pi : D^{1\times p} \longrightarrow M$ (resp., $\pi' : D^{1\times p'} \longrightarrow M'$) the canonical projection.*

1. *The existence of $f \in \hom_D(M, M')$ is equivalent to the existence of two matrices $P \in D^{p\times p'}$ and $Q \in D^{q\times q'}$ satisfying the following identity:*

$$R\,P = Q\,R'.$$

Then, $f \in \hom_D(M, M')$ is defined by $f(\pi(\lambda)) = \pi'(\lambda\, P)$ for all $\lambda \in D^{1\times p}$, and we have the following commutative exact diagram

$$\begin{tikzcd} D^{1\times q} \arrow[r, ".R"] \arrow[d, ".Q"] & D^{1\times p} \arrow[r, "\pi"] \arrow[d, ".P"] & M \arrow[r] \arrow[d, "f"] & 0 \\ D^{1\times q'} \arrow[r, ".R'"] & D^{1\times p'} \arrow[r, "\pi'"] & M' \arrow[r] & 0, \end{tikzcd}$$

namely, every square commutes, i.e., $.P \circ .R = .R' \circ .Q$ and $f \circ \pi = \pi' \circ .P$.

2. *Let $R'_2 \in D^{q'_2\times q'}$ be such that $\ker_D(.R') = \operatorname{im}_D(.R'_2)$ and let $P \in D^{p\times p'}$ and $Q \in D^{q\times q'}$ be two matrices satisfying $R\, P = Q\, R'$. Then, the following matrices*

$$\overline{P} := P + Z\, R', \quad \overline{Q} := Q + R\, Z + Z_2\, R'_2,$$

where $Z \in D^{p\times q'}$ and $Z_2 \in D^{q\times q'_2}$ are two arbitrary matrices, satisfy the relation $R\,\overline{P} = \overline{Q}\, R'$ and $f(\pi(\lambda)) = \pi'(\lambda\, P) = \pi'(\lambda\,\overline{P})$ for all $\lambda \in D^{1\times p}$.

For algorithms to compute the matrices P and Q for different classes of linear functional systems, we refer to [6] and the OREMORPHISMS and OREALGEBRAICANALYSIS packages [7, 12].

Example 4. We consider again Example 3. Let $M' := D/(D^{1\times 2}\, R')$ be the $D := \mathbb{Q}[\partial, \delta]$-module finitely presented by the matrix $R' := (\partial^2 - \delta^2\,(\delta+1) \quad \partial+1)^T$, which corresponds to the following linear DTD system:

$$\begin{cases} \ddot{z}(t) - z(t-3\,h) - z(t-2\,h) = 0, \\ \dot{z}(t) + z(t) = 0. \end{cases} \tag{8}$$

Let $\pi' : D \longrightarrow M'$ be the canonical projection. A homomorphism $f : M \longrightarrow M'$ is defined by $f(\pi(\lambda)) = \pi'(\lambda\, P)$ for all $\lambda \in D^{1\times 2}$, where

$$P := \begin{pmatrix} 1 \\ \partial \end{pmatrix}, \quad Q := \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix},$$

since we can easily check that we have $R\, P = Q\, R'$.

Coming back to $\eta := (\phi'(f(y_1)) \ \dots \ \phi'(f(y_p)))^T$, using (6), we get

$$\eta_j := \phi'(f(y_j)) = \phi'\left(\sum_{k=1}^{p'} P_{jk}\, y'_k\right) = \sum_{k=1}^{p'} P_{jk}\, \phi'(y'_k) = \sum_{k=1}^{p'} P_{jk}\, \eta'_k,$$

which shows that $\eta := P\,\eta' \in \ker_{\mathcal{F}}(R.)$ for all $\eta' \in \ker_{\mathcal{F}}(R'.)$.

Corollary 1. *With the notations of Lemma 1, if $\mathcal{F}$ is a left D-module, then we have:*

$$\begin{array}{rcl} P. : \ker_{\mathcal{F}}(R'.) & \longrightarrow & \ker_{\mathcal{F}}(R.) \\ \eta' & \longmapsto & \eta := P\,\eta'. \end{array} \tag{9}$$

The *contravariant functor* $\hom_D(\cdot\,, \mathcal{F})$ (see, e.g., [24]) transforms finitely presented left D-modules (resp., homomorphisms of finitely presented left D-modules) into $\mathcal{F}$-behaviors (resp., homomorphisms between $\mathcal{F}$-behaviors in the reverse direction).

Example 5. We consider again Examples 3 and 4. Using $f \in \hom_D(M, M')$, we have (9), where $P := (1 \quad \partial)^T$, i.e., the additive mapping

$$z(t) \longmapsto \begin{cases} x_1(t) = z(t), \\ x_2(t) = \dot{z}(t), \end{cases} \tag{10}$$

sends $\mathcal{F}$-solutions of (8) to $\mathcal{F}$-solutions of (5), where $\mathcal{F}$ is a $D := \mathbb{Q}[\partial, \delta]$-module.

Let $f : M \longrightarrow M'$ be a homomorphism of left/right D-modules. Then, we can define the kernel, image, coimage, and cokernel of f as the following left/right D-modules:

$$\begin{array}{ll} \ker f := \{m \in M \mid f(m) = 0\}, & \operatorname{im} f := \{m' \in M' \mid \exists\, m \in M : \; m' = f(m)\}, \\ \operatorname{coim} f := M / \ker f, & \operatorname{coker} f := M' / \operatorname{im} f. \end{array}$$

Finally, let us explicitly characterize the latter modules.

Lemma 2 ([6])**.** *Let $M := D^{1\times p}/(D^{1\times q}\, R)$ (resp., $M' := D^{1\times p'}/(D^{1\times q'}\, R')$) be the left D-module finitely presented by $R \in D^{q\times p}$ (resp., $R' \in D^{q'\times p'}$). Moreover, let $f \in \hom_D(M, M')$ be defined by $P \in D^{p\times p'}$ and $Q \in D^{q\times q'}$ satisfying (7).*

1. *Let $S \in D^{r\times p}$ and $T \in D^{r\times q'}$ be two matrices such that*

$$\ker_D(.(P^T \quad R'^T)^T) = \operatorname{im}_D(.(S \quad -T)),$$

$L \in D^{q\times r}$ a matrix satisfying $R = L\, S$ and a matrix $S_2 \in D^{r_2\times r}$ such that $\ker_D(.S) = \operatorname{im}_D(.S_2)$. Then, we have:

$$\ker f = (D^{1\times r}\, S)/(D^{1\times q}\, R) \cong D^{1\times r} / \left(D^{1\times(q+r_2)} \begin{pmatrix} L \\ S_2 \end{pmatrix} \right).$$

2. *With the above notations, we have:*

$$\operatorname{coim} f = D^{1\times p}/(D^{1\times r}\, S) \cong \operatorname{im} f = \left(D^{1\times(p+q')} \begin{pmatrix} P \\ R' \end{pmatrix} \right) / (D^{1\times q'}\, R').$$

3. *We have $\operatorname{coker} f = D^{1\times p'} / \left(D^{1\times(p+q')} \begin{pmatrix} P^T & R'^T \end{pmatrix}^T \right)$. Thus, $\operatorname{coker} f$ admits the following beginning of a finite free resolution:*

$$D^{1\times r} \xrightarrow{.(S \quad -T)} D^{1\times(p+q')} \xrightarrow{.\begin{pmatrix} P \\ R' \end{pmatrix}} D^{1\times p'} \xrightarrow{\epsilon} \operatorname{coker} f \longrightarrow 0. \tag{11}$$

4. *We have the following commutative exact diagram*

$$\begin{tikzcd} & & 0 \arrow[d] & \\ D^{1\times r} \arrow[r, ".S"] \arrow[d, ".T"] & D^{1\times p} \arrow[r, "\kappa"] \arrow[d, ".P"] & \operatorname{coim} f \arrow[r] \arrow[d, "f^\sharp"] & 0 \\ D^{1\times q'} \arrow[r, ".R'"] & D^{1\times p'} \arrow[r, "\pi'"] & M' \arrow[r] \arrow[d] & 0, \\ & & \operatorname{coker} f \arrow[d] & \\ & & 0 & \end{tikzcd}$$

where $f^\sharp : \operatorname{coim} f \longrightarrow M'$ is defined by $f^\sharp(\kappa(\lambda)) = \pi'(\lambda\, P)$ for all $\lambda \in D^{1\times p}$.

We note that $M := D^{1\times p}/(D^{1\times q}\, R)$ is the zero module if and only if we have $D^{1\times q}\, R = D^{1\times p}$, i.e., if and only if there exists a matrix $T \in D^{p\times q}$ such that $T\, R = I_p$, i.e., if and only if the presentation matrix R of M admits a *left inverse*. Using this result and Lemma 2, we can now characterize when $f \in \hom_D(M, M')$ is the zero homomorphism, injective, surjective or defines an isomorphism.

Lemma 3 ([6])**.** *With the notations of Lemma 2, $f \in \hom_D(M, M')$ is:*

1. *The zero homomorphism, i.e., $f = 0$, if and only if one of the following equivalent conditions holds:*

   (a) *There exists $Z \in D^{p \times q'}$ such that $P = Z\,R'$. If $R'_2 \in D^{q'_2 \times q'}$ is a matrix satisfying $\ker_D(.R') = \operatorname{im}_D(.R'_2)$, then there exists $Z_2 \in D^{q \times q'_2}$ such that:*

   $$Q = R\,Z + Z_2\,R'_2.$$

   (b) *The matrix S admits a left inverse, i.e., there exists $X \in D^{p \times r}$ such that:*

   $$X\,S = I_p.$$

2. *Injective, i.e., $\ker f = 0$, if and only if one of the following equivalent conditions holds:*

   (a) *There exists $F \in D^{r \times q}$ such that $S = F\,R$. Then, if $\rho : M \longrightarrow \operatorname{coim} f$ is the canonical projection, then we have $f = f^{\sharp} \circ \rho$, where $f^{\sharp} \in \hom_D(\operatorname{coim} f, M')$ is defined in 4 of Lemma 2, and the following commutative exact diagram shows that ρ is an isomorphism:*

   $$\begin{tikzcd}
   & 0 & 0 & \\
   D^{1\times q} \arrow[r, ".R"] & D^{1\times p} \arrow[r, "\pi"] \arrow[u] & M \arrow[r] \arrow[u] & 0 \\
   D^{1\times r} \arrow[r, ".S"] \arrow[u, ".F"] & D^{1\times p} \arrow[r, "\kappa"] \arrow[u, equal] & \operatorname{coim} f \arrow[r] \arrow[u, "\rho^{-1}"] & 0. \\
   & 0 \arrow[u] & 0 \arrow[u] &
   \end{tikzcd}$$

   (b) *The matrix $(L^T \quad S_2^T)^T$ admits a left inverse.*

3. *Surjective, i.e., $\operatorname{im} f = M'$, if and only if $(P^T \quad R'^T)^T$ admits a left inverse. Then, the long exact sequence (11)* splits *(see, e.g., [24]), i.e., there exist four matrices $P' \in D^{p' \times p}$, $Z' \in D^{p' \times q'}$, $U \in D^{p \times r}$, and $V \in D^{q' \times r}$ such that:*

   $$\begin{cases} P'\,P + Z'\,R' = I_{p'}, \\ P\,P' + U\,S = I_p, \\ P\,Z' - U\,T = 0, \\ R'\,P' - V\,S = 0, \\ R'\,Z' + V\,T = I_{q'}. \end{cases}$$

   In this case, we have the following commutative exact diagram:

   $$\begin{tikzcd}
   & & 0 & \\
   D^{1\times r} \arrow[r, ".S"] & D^{1\times p} \arrow[r, "\kappa"] & \operatorname{coim} f \arrow[r] \arrow[u] & 0 \\
   D^{1\times q'} \arrow[r, ".R'"] \arrow[u, ".V"] & D^{1\times p'} \arrow[r, "\pi'"] \arrow[u, ".P'"] & M' \arrow[r] \arrow[u, "f^{\sharp\,-1}"] & 0. \\
   & & 0 \arrow[u] &
   \end{tikzcd}$$

4. *An isomorphism, i.e., $M \cong M'$, if and only if both matrices $(L^T \quad S_2^T)^T$ and $(P^T \quad R'^T)^T$ admit a left inverse. The inverse f^{-1} of f is then defined by*

   $$\forall\, \lambda' \in D^{1\times p'}, \quad f^{-1}(\pi'(\lambda')) := \pi(\lambda'\,P'),$$

where $P' \in D^{p' \times p}$ is a matrix as defined in 3. Moreover, we have the following commutative exact diagram:

$$\begin{tikzcd} D^{1\times q} \arrow[r, ".R"] & D^{1\times p} \arrow[r, "\pi"] & M \arrow[r] & 0 \\ D^{1\times q'} \arrow[u, ".V\,F"] \arrow[r, ".R'"] & D^{1\times p'} \arrow[u, ".P'"] \arrow[r, "\pi'"] & M' \arrow[u, "f^{-1}"] \arrow[r] & 0. \end{tikzcd}$$

Algorithms for checking whether or not a homomorphism of finitely presented left D-modules is injective, surjective, or defines an isomorphism (and if so, compute its inverse) are implemented in the OREMORPHISMS package [7].

Example 6. Let us check that the homomorphism f defined in Example 4 is an isomorphism by characterizing $\ker f$ and $\operatorname{coker} f$, and then let us explicitly compute its inverse f^{-1}. Using elimination techniques, we can first check that f is surjective, i.e., $\operatorname{coker} f = 0$, since $(P' \quad Z') := (0 \quad -1 \quad 0 \quad 1)$ is a left inverse of the matrix $(P^T \quad R'^T)^T$. We also have $\ker_D(.(P^T \quad R'^T)^T) = \operatorname{im}_D(.(S \quad -T))$, where:

$$S := \begin{pmatrix} 1 & 1 \\ 0 & \partial + 1 \\ 0 & \delta^2\,(\delta+1) - 1 \\ 0 & 0 \end{pmatrix}, \quad T := \begin{pmatrix} 0 & 1 \\ 0 & \partial \\ -\partial & \partial\,(\partial - 1) \\ -\partial - 1 & \partial^2 - \delta^2\,(\delta+1) \end{pmatrix}.$$

Moreover, the identities of 3. of Lemma 3 are satisfied with:

$$U := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad V := \begin{pmatrix} 0 & -\partial + 1 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

Let us now check that f is injective. We have $R = L\,S$ and $\ker_D(.S) = \operatorname{im}_D(.S_2)$, where:

$$L := \begin{pmatrix} \partial & -1 & 0 & 0 \\ -\delta^2\,(\delta+1) & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad S_2 := \begin{pmatrix} 0 & \delta^2\,(\delta+1) - 1 & -\partial - 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The matrix $(L^T \quad S_2^T)^T$ admits the following left inverse defined by

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & \partial & 0 & 0 \\ 1 & 1 & \delta^2\,(\delta+1) - \partial & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

which shows that $\ker f = 0$ and proves that f is an isomorphism, i.e., $M \cong M'$. Hence, for every D-module $\mathcal{F}$, we get $\ker_{\mathcal{F}}(R.) \cong \ker_{\mathcal{F}}(R'.)$, i.e., there exists a 1-1 correspondence between the $\mathcal{F}$-solutions of (5) and the $\mathcal{F}$-solutions of (8) or, in other words, the linear DTD systems (5) and (8) are equivalent. More precisely, using 4 of Lemma 3, we obtain that $f^{-1} : M' \longrightarrow M$ is defined by $f^{-1}(\pi'(\lambda')) := \pi(\lambda'\,P')$, where $P' := (0 \quad -1)$. In terms of behaviors, the following homomorphism

$$\begin{array}{rcl} P'. : \ker_{\mathcal{F}}(R.) & \longrightarrow & \ker_{\mathcal{F}}(R'.) \\ \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} & \longmapsto & z(t) := -x_2(t), \end{array}$$

is the inverse of the homomorphism of behaviors $P.$ defined by (10).

4 Characterization of isomorphic modules

We characterize the existence of a left/right/two sided inverse of a homomorphism.

Lemma 4. *With the notations of Lemma 2, we have:*

1. *f admits a right inverse $g \in \hom_D(M', M)$, i.e., $f \circ g = \mathrm{id}_{M'}$, or equivalently we have $M \cong \ker f \oplus M'$, if and only if there exist three matrices $P' \in D^{p' \times p}$, $Q' \in D^{q' \times q}$, and $Z' \in D^{p' \times q'}$ satisfying:*
$$R'\,P' = Q'\,R, \quad P'\,P + Z'\,R' = I_{p'}.$$
Then, for any matrix $R'_2 \in D^{r' \times q'}$ such that $\ker_D(.R') = \mathrm{im}_D(.R'_2)$, there exists $Z'_2 \in D^{q' \times r'}$ satisfying $Q'\,Q + R'\,Z' + Z'_2\,R'_2 = I_{q'}$.

2. *f admits a left inverse $g \in \hom_D(M', M)$, i.e., $g \circ f = \mathrm{id}_M$, or equivalently we have $M' \cong M \oplus \mathrm{coker}\, f$, if and only if there exist three matrices $P' \in D^{p' \times p}$, $Q' \in D^{q' \times q}$ and $Z \in D^{p \times q}$ satisfying:*
$$R'\,P' = Q'\,R, \quad P\,P' + Z\,R = I_p.$$
Then, for any matrix $R_2 \in D^{r \times q}$ such that $\ker_D(.R) = \mathrm{im}_D(.R_2)$, there exists $Z_2 \in D^{q \times r}$ satisfying $Q\,Q' + R\,Z + Z_2\,R_2 = I_q$.

3. *f is an isomorphism, and thus $M \cong M'$, if and only if there exist 4 matrices $P' \in D^{p' \times p}$, $Q' \in D^{q' \times q}$, $Z \in D^{p \times q}$, and $Z' \in D^{p' \times q'}$ satisfying:*
$$R'\,P' = Q'\,R, \quad P\,P' + Z\,R = I_p, \quad P'\,P + Z'\,R' = I_{p'}. \tag{12}$$
Then, for $R_2 \in D^{r \times q}$ (resp., $R'_2 \in D^{r' \times q'}$) such that $\ker_D(.R) = \mathrm{im}_D(.R_2)$ (resp., $\ker_D(.R') = \mathrm{im}_D(.R'_2)$), there exist matrices $Z_2 \in D^{q \times r}$, $Z'_2 \in D^{q' \times r'}$, $Y_2 \in D^{p' \times r}$, $Y'_2 \in D^{p \times r'}$ such that:
$$\begin{aligned} &Q\,Q' + R\,Z + Z_2\,R_2 = I_q, && Q'\,Q + R'\,Z' + Z'_2\,R'_2 = I_{q'}, \\ &Z'\,Q' - P'\,Z = Y_2\,R_2, && P\,Z' - Z\,Q = Y'_2\,R'_2. \end{aligned} \tag{13}$$

Proof. 1. The existence of $g \in \hom_D(M', M)$ is equivalent to the existence of two matrices $P' \in D^{p' \times p}$ and $Q' \in D^{q' \times q}$ such that $R'\,P' = Q'\,R$ (see 1 of Lemma 1). Composing the following two commutative exact diagrams

$$\begin{tikzcd} D^{1\times q} \arrow[r, ".R"] \arrow[d, ".Q"] & D^{1\times p} \arrow[r, "\pi"] \arrow[d, ".P"] & M \arrow[r] \arrow[d, "f"] & 0 \\ D^{1\times q'} \arrow[r, ".R'"] & D^{1\times p'} \arrow[r, "\pi'"] & M' \arrow[r] & 0, \end{tikzcd} \qquad \begin{tikzcd} D^{1\times q} \arrow[r, ".R"] & D^{1\times p} \arrow[r, "\pi"] & M \arrow[r] & 0 \\ D^{1\times q'} \arrow[r, ".R'"] \arrow[u, ".Q'"] & D^{1\times p'} \arrow[r, "\pi'"] \arrow[u, ".P'"] & M' \arrow[r] \arrow[u, "g"] & 0, \end{tikzcd}$$

and noting $\chi := \mathrm{id}_{M'} - f \circ g$, we get the following commutative exact diagram:

$$\begin{tikzcd} D^{1\times q'} \arrow[r, ".R'"] & D^{1\times p'} \arrow[r, "\pi'"] & M' \arrow[r] & 0 \\ D^{1\times q'} \arrow[r, ".R'"] \arrow[u, ".(I_{q'}-Q'\,Q)"] & D^{1\times p'} \arrow[r, "\pi'"] \arrow[u, ".(I_{p'}-P'\,P)"] & M' \arrow[r] \arrow[u, "\chi"] & 0. \end{tikzcd}$$

By 1.a of Lemma 3, $\chi = 0$ if and only if there exists $Z' \in D^{p' \times q'}$ such that $I_{p'} - P'\,P = Z'\,R'$, i.e., $P'\,P + Z'\,R' = I_{p'}$. According to 1.a of Lemma 3, there then exists a matrix $Z'_2 \in D^{q' \times r'}$ satisfying the relation $I_{q'} - Q'\,Q = R'\,Z' + Z'_2\,R'_2$, i.e., $Q'\,Q + R'\,Z' + Z'_2\,R'_2 = I_{q'}$, where $R'_2 \in D^{r' \times q'}$ is such that $\ker_D(.R') = \mathrm{im}_D(.R'_2)$. Finally, $M \cong \ker f \oplus M'$ is well-known to be equivalent to the *splitting* of the following short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \ker f \arrow[r] & M \arrow[r, shift left, "f"] & M' \arrow[l, shift left, "g"] \arrow[r] & 0, \end{tikzcd}$$

(see, e.g., [24]), i.e., it is equivalent to the existence of a left inverse g of f.

2 can be proved similarly as 1. The first points of 3 are direct consequences of 1 and 2. Finally, let us prove the third and fourth identities of (13). Using the identity $Q'\,R = R'\,P'$ and (12), we have

$$(Z'\,Q' - P'\,Z)\,R = (Z'\,R')\,P' - P'\,(Z\,R) = (I_{p'} - P'\,P)\,P' - P'\,(I_p - P\,P') = 0,$$

which yields $\mathrm{im}_D(.(Z'\,Q' - P'\,Z)) \subseteq \ker_D(.R) = \mathrm{im}_D(.R_2)$ and shows that there exists $Y_2 \in D^{p' \times r}$ such that $Z'\,Q' - P'\,Z = Y_2\,R_2$. Similarly, using $Q\,R' = R\,P$ and (12), we have

$$(P\,Z' - Z\,Q)\,R' = P\,(Z'\,R') - (Z\,R)\,P = P\,(I_{p'} - P'\,P) - (I_p - P\,P')\,P = 0,$$

which yields $\mathrm{im}_D(.(P\,Z' - Z\,Q)) \subseteq \ker_D(.R') = \mathrm{im}_D(.R_2')$ and shows that there exists $Y_2' \in D^{p \times r'}$ such that $P\,Z' - Z\,Q = Y_2'\,R_2'$.

$\square$

Remark 2. We note that the existence of a right (resp., left) inverse g of $f \in \hom_D(M, M')$ implies that f is surjective (resp., injective) since we then have $m' = f(g(m'))$ (resp., $g(f(m)) = m$) for all $m' \in M'$ (resp., $m \in M$).

Example 7. We can check again that the D-module M and M' defined in Examples 3 and 4 are isomorphic by considering the matrices P and Q defined in Example 4 and the matrix P' defined in Example 6. Then, we have:

$$Q' := \begin{pmatrix} \partial & 1 & -\partial^2 + \delta^2\,(\delta + 1) \\ 1 & 0 & -\partial \end{pmatrix}, \quad Z := \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & \partial \end{pmatrix}, \quad Z' := \begin{pmatrix} 0 & 1 \end{pmatrix}.$$

We can check that we have $Q\,Q' + R\,Z = I_3$, $Q'\,Q + R'\,Z' = I_2$, $Z'\,Q' - P'\,Z = 0$, and $P\,Z' - Z\,Q = 0$, i.e., $Z_2 = 0$, $Z_2' = 0$, $Y_2 = 0$, and $Y_2' = 0$.

Let us introduce a few definitions.

Definition 1.

1. We denote the *general linear group of degree* r over D by:

$$\mathrm{GL}_r(D) := \{U \in D^{r \times r} \mid \exists\, V \in D^{r \times r} : U\,V = V\,U = I_r\}.$$

2. Two matrices R, $R' \in D^{q \times p}$ are said to be *equivalent* if there exist $U \in \mathrm{GL}_q(D)$ and $V \in \mathrm{GL}_p(D)$ such that:

$$R' = U\,R\,V.$$

In module theory, *Fitting's theorem* states that two finitely presented modules are isomorphic if and only if their presentation matrices R and R' can be inflated by zero and identity matrices in a way that the new matrices are equivalent. More precisely, Fitting's theorem states that $M := D^{1 \times p}/(D^{1 \times q}\,R) \cong M' := D^{1 \times p'}/(D^{1 \times q'}\,R')$ if and only if the following two matrices

$$L := \begin{pmatrix} R & 0 \\ 0 & I_{p'} \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \, L' := \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ I_p & 0 \\ 0 & R' \end{pmatrix} \in D^{(q+p'+p+q') \times (p+p')},$$

are equivalent. For a constructive version of Fitting's theorem, see [8].

In this paper, we give another characterization of isomorphic finitely presented modules in terms of inflations of their presentation matrices.

Theorem 1. *Let $R \in D^{q \times p}$ and $R' \in D^{q' \times p'}$ be two matrices with entries in a noetherian ring D. Then, the following assertions are equivalent:*

1. $M := D^{1 \times p}/(D^{1 \times q}\,R) \cong M' := D^{1 \times p'}/(D^{1 \times q'}\,R')$.

2. *There exist 12 matrices*

$$P \in D^{p \times p'},\ Q \in D^{q \times q'},\ P' \in D^{p' \times p},\ Q' \in D^{q' \times q},\ Z \in D^{p \times q},\ Z' \in D^{p' \times q'},$$
$$Z_2 \in D^{q \times r},\ Y_2 \in D^{p' \times r},\ Y_2' \in D^{p \times r'},\ Z_2' \in D^{q' \times r'},\ R_2 \in D^{r \times q},\ R_2' \in D^{r' \times q'}$$

satisfying the following two identities

$$\begin{pmatrix} R & -Q \\ P' & Z' \end{pmatrix} \begin{pmatrix} Z & P \\ -Q' & R' \end{pmatrix} + \begin{pmatrix} Z_2 \\ Y_2 \end{pmatrix} (R_2 \quad 0) \quad = I_{q+p'}, \tag{14}$$

$$\begin{pmatrix} Z & P \\ -Q' & R' \end{pmatrix} \begin{pmatrix} R & -Q \\ P' & Z' \end{pmatrix} + \begin{pmatrix} Y_2' \\ Z_2' \end{pmatrix} (0 \quad R_2') \quad = I_{p+q'}, \tag{15}$$

where the matrices $R_2 \in D^{r \times q}$ and $R_2' \in D^{r' \times q'}$ are such that:

$$\ker_D(.R) = \operatorname{im}_D(.R_2), \quad \ker_D(.R') = \operatorname{im}_D(.R_2').$$

Proof. By 3 of Lemma 4, $M \cong M'$ if and only if there exist $P' \in D^{p' \times p}$, $Q' \in D^{q' \times q}$, $Z \in D^{p \times q}$, and $Z' \in D^{p' \times q'}$ satisfying (12). Using (12) and (13), we get

$$\begin{aligned}
\begin{pmatrix} R & -Q \\ P' & Z' \end{pmatrix} \begin{pmatrix} Z & P \\ -Q' & R' \end{pmatrix} &= \begin{pmatrix} I_q - Z_2\, R_2 & 0 \\ -Y_2\, R_2 & I_{p'} \end{pmatrix} \\
&= I_{q+p'} - \begin{pmatrix} Z_2 \\ Y_2 \end{pmatrix} (R_2 \quad 0), \\
\begin{pmatrix} Z & P \\ -Q' & R' \end{pmatrix} \begin{pmatrix} R & -Q \\ P' & Z' \end{pmatrix} &= \begin{pmatrix} I_p & -Y_2'\, R_2' \\ 0 & I_{q'} - Z_2'\, R_2' \end{pmatrix} \\
&= I_{p+q'} - \begin{pmatrix} Y_2' \\ Z_2' \end{pmatrix} (0 \quad R_2'),
\end{aligned}$$

i.e., (14) and (15) hold. Conversely, if (14) and (15) hold, then we have $R\,P = Q\,R'$, $R'\,P' = Q'\,R$, $P\,P' + Z\,R = I_p$, and $P'\,P + Z'\,R' = I_{p'}$, which shows that $M \cong M'$ by 3 of Lemma 4. □

Example 8. We consider again Examples 3 and 4. We first can check that we have $\ker_D(.R) = \operatorname{im}_D(.R_2)$ and $\ker_D(.R') = \operatorname{im}_D(.R_2')$, where:

$$\begin{cases} R_2 := \left(\partial + \delta^2\,(\delta+1) \quad \partial + 1 \quad -\partial^2 + \delta^2\,(\delta+1)\right), \\ R_2' := \left(\partial + 1 \quad -\partial^2 + \delta^2\,(\delta+1)\right). \end{cases}$$

Hence, R and R' are not full row rank matrices. Using Example 7, Theorem 1 yields:

$$\begin{pmatrix} \partial & -1 & 0 & 0 \\ -\delta^2\,(\delta+1) & \partial & -1 & 0 \\ 1 & 1 & 0 & -1 \\ 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 1 \\ -1 & 0 & \partial & \partial \\ -\partial & -1 & \partial^2 - \delta^2\,(\delta+1) & \partial^2 - \delta^2\,(\delta+1) \\ -1 & 0 & \partial & \partial + 1 \end{pmatrix} = I_4.$$

Note that (15) is a consequence of the above identity since D is a commutative ring.

Example 9. We consider the following linear system of PDEs

$$\begin{cases} \dfrac{\partial^2 y(x_1,x_2)}{\partial x_1^2} - x_2\,\dfrac{\partial^2 y(x_1,x_2)}{\partial x_2^2} - \dfrac{\beta}{2}\,\dfrac{\partial y(x_1,x_2)}{\partial x_2} = 0, \\ 2\,\dfrac{\partial^2 y(x_1,x_2)}{\partial x_1 \partial x_2} + x_1\,\dfrac{\partial^2 y(x_1,x_2)}{\partial x_2^2} = 0, \end{cases} \tag{16}$$

which is studied in probability theory [2]. Let $D := \mathbb{Q}(\beta)(x_1,x_2)\langle \partial_1, \partial_2 \rangle$ be the noncommutative ring of PD operators in $\partial_1 := \frac{\partial}{\partial x_1}$ and $\partial_2 := \frac{\partial}{\partial x_2}$ with coefficients in the field $\mathbb{Q}(\beta, x_1, x_2)$ of rational functions in x_1, x_2, and β. The ring D is called the *Weyl algebra* in two variables and it is usually denoted by $B_2(\mathbb{Q}(\beta))$. Let us consider the matrix of PD operators associated with (16)

$$R := \begin{pmatrix} \partial_1^2 - x_2\,\partial_2^2 - \frac{\beta}{2}\,\partial_2 \\ 2\,\partial_1\,\partial_2 + x_1\,\partial_2^2 \end{pmatrix} \in D^{2\times 1},$$

and the left D-module $M := D/(D^{1\times 2}\,R)$ finitely presented by R. It can be shown that M is *D-finite*, namely, M has a $\mathbb{Q}(\beta, x_1, x_2)$-finite dimensional vector space structure (see, e.g., [5]), and thus it can be written as an *integrable connection*, i.e., we can find a first-order realization of (16) (see, e.g., [6]). We can show that (16), i.e., $R\,\eta = 0$, is equivalent to $R'\,\eta' = 0$, where

$$R' := \begin{pmatrix} \partial_1 & 0 & -1 & 0 \\ 0 & \partial_1 & 0 & \frac{1}{2}\,x_1 \\ 0 & -\frac{\beta}{2} & \partial_1 & -x_2 \\ 0 & 0 & 0 & \partial_1 + \frac{(\beta+3)x_1}{{x_1}^2 - 4\,x_2} \\ \partial_2 & -1 & 0 & 0 \\ 0 & \partial_2 & 0 & -1 \\ 0 & 0 & \partial_2 & \frac{1}{2}\,x_1 \\ 0 & 0 & 0 & \partial_2 - \frac{2\,\beta+6}{{x_1}^2-4\,x_2} \end{pmatrix} \in D^{8\times 4},$$

i.e., we have $M \cong M' := D^{1\times 4}/(D^{1\times 8}\,R')$. This first-order realization can be computed by means of the OREMODULES package [4]. Let us compute the matrices appearing in (14) and (15). By construction, the isomorphism $f \in \hom_D(M, M')$ is defined by $f(\pi(\lambda)) := \pi'(\lambda\,P)$, where $\pi : D \longrightarrow M$ (resp., $\pi' : D^{1\times 4} \longrightarrow M'$) is the canonical projection, $\lambda \in D$, and $P := (1 \quad 0 \quad 0 \quad 0)$ satisfies (7), where the matrix Q is defined by:

$$Q := \begin{pmatrix} \partial_1 & 0 & 1 & 0 & -x_2\,\partial_2 - \frac{\beta}{2} & -x_2 & 0 & 0 \\ 0 & 2 & 0 & 0 & x_1\,\partial_2 + 2\,\partial_1 & x_1 & 0 & 0 \end{pmatrix} \in D^{2\times 8}.$$

Since f is surjective, the matrix $(P^T \quad R'^T)^T$ admits a left inverse $(P' \quad Z')$, where $P' := \begin{pmatrix} 1 & \partial_2 & \partial_1 & \partial_2^2 \end{pmatrix}^T$ and:

$$Z' := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\partial_2 & -1 & 0 & 0 \end{pmatrix} \in D^{4\times 8}.$$

Moreover, we have $R'\,P' = Q'\,R$, where the matrix Q' is defined by

$$Q' := \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2} \\ 1 & 0 \\ -\frac{2\,x_1\,\partial_2}{{x_1}^2-4\,x_2} & \frac{-2\,x_2\,\partial_2 + x_1\,\partial_1}{{x_1}^2-4\,x_2} \\ 0 & 0 \\ 0 & 0 \\ 0 & \frac{1}{2} \\ \frac{4\,\partial_2}{{x_1}^2-4\,x_2} & \frac{-2\,\partial_1 + x_1\,\partial_2}{{x_1}^2-4\,x_2} \end{pmatrix} \in D^{8\times 2},$$

and $f^{-1}(\pi'(\lambda')) := \pi(\lambda'\,P')$ for all $\lambda' \in D^{1\times 4}$.

Using (12), we can check that $P\,P' = 1$, i.e., $Z = 0$, and $Q\,Q' = I_2$, i.e., $Z_2 = 0$ (see (13)). We also have

$\ker_D(.R) = \operatorname{im}_D(.R_2)$ and $\ker_D(.R') = \operatorname{im}_D(.R_2')$, where

$$R_2 := (2\,x_1\,\partial_2^2 + 4\,\partial_1\,\partial_2 \quad -2\,\partial_1^2 + 2\,x_2\,\partial_2^2 + (4+\beta)\,\partial_2),$$

$$R_2' :=$$
$$\left(\begin{array}{cccccccc} -\partial_2 & 1 & 0 & 0 & \partial_1 & 0 & -1 & 0 \\ 0 & -4\,x_2\,\partial_2 & -2\,x_1\,\partial_2 & 4\,x_2 - {x_1}^2 & 0 & -x_1\,\beta + 4\,x_2\,\partial_1 & 2\,\partial_1\,x_1 & 0 \\ 0 & -2\,x_1\,\partial_2 & -4\,\partial_2 & 0 & 0 & 2\,x_1\,\partial_1 - 2\,\beta & 4\,\partial_1 & -4\,x_2 + {x_1}^2 \\ 0 & 2\,(\beta+1)\,\partial_2 & 0 & (4\,x_2 - {x_1}^2)\,\partial_2 + 4 & 0 & -2\,(\beta+1)\,\partial_1 & 0 & ({x_1}^2 - 4\,x_2)\,\partial_1 + 2\,x_1 \end{array}\right),$$

and, using (13), we get:

$$Z_2' = \left(\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{2\,x_1\,\partial_1}{{x_1}^2 - 4\,x_2} & -\frac{1}{{x_1}^2 - 4\,x_2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ \frac{4\,\partial_1}{{x_1}^2 - 4\,x_2} & 0 & \frac{1}{x_1^2 - 4\,x_2} & 0 \end{array}\right) \in D^{8\times 4}.$$

Using (13) again, we get $Y_2 = 0$ and $Y_2' = 0$, and thus we finally obtain:

$$\left(\begin{array}{cc} R & -Q \\ P' & Z' \end{array}\right) \left(\begin{array}{cc} Z & P \\ -Q' & R' \end{array}\right) = I_6, \quad \left(\begin{array}{cc} Z & P \\ -Q' & R' \end{array}\right) \left(\begin{array}{cc} R & -Q \\ P' & Z' \end{array}\right) + \left(\begin{array}{c} 0 \\ Z_2' \end{array}\right) (0 \quad R_2') = I_9.$$

For applications of D-finite multidimensional systems in control theory, see [20].

A consequence of Theorem 1 connects isomorphisms of finitely presented modules to the *unimodular completion problem*, and therefore to the so-called *Serre's reduction problem* studied in [1, 9, 11] (see Sections 1 and 5).

Corollary 2. *With the notations and the assumptions of Theorem 1, let us assume that we have $q + p' = p + q'$.*

1. *Then, we have:*

$$\left(\begin{array}{cc} R & -Q \\ P' & Z' \end{array}\right) \left(\begin{array}{cc} Z & P \\ -Q' & R' \end{array}\right) = I_{q+p'} \iff \left(\begin{array}{cc} Z & P \\ -Q' & R' \end{array}\right) \left(\begin{array}{cc} R & -Q \\ P' & Z' \end{array}\right) = I_{p+q'}.$$

2. *If R or R' have full row rank, then the fact that $M \cong M'$ is equivalent to the existence of matrices $P \in D^{p\times p'}$, $Q \in D^{q\times q'}$, $P' \in D^{p'\times p}$, $Q' \in D^{q'\times q}$, $Z \in D^{p\times q}$, and $Z' \in D^{p'\times q'}$ such that:*

$$\left(\begin{array}{cc} R & -Q \\ P' & Z' \end{array}\right) \left(\begin{array}{cc} Z & P \\ -Q' & R' \end{array}\right) = I_{q+p'}.$$

Proof. 1 is a consequence of $q + p' = p + q'$ and the fact that D is a noetherian ring, and thus a *stably finite ring*, namely, a ring for which $U\,V = I_r$ for two matrices $U,\, V \in D^{r\times r}$ yields $V\,U = I_r$ (see, e.g., [16, 24]). Note that a commutative ring is stably finite since $U\,V = I_r$ implies that $\det U$ is a unit of D.

2 is a direct consequence of 1 and Theorem 1 with $R_2 = 0$ or $R_2' = 0$. □

Example 10. Let $R,\, R' \in D^{q\times p}$ be two equivalent matrices, i.e., they satisfy

$$R' = Q^{-1}\,R\,P,$$

for certain $P \in \mathrm{GL}_p(D)$ and $Q \in \mathrm{GL}_q(D)$. If we note $M := D^{1\times p}/(D^{1\times q}\,R)$ and $M' := D^{1\times p}/(D^{1\times q}\,R')$, then $f \in \hom_D(M, M')$, defined by $f(\pi(\lambda)) := \pi'(\lambda\,P)$ for all $\lambda \in D^{1\times p}$, is an isomorphism and $f^{-1} \in$

$\hom_D(M', M)$ is defined by $f^{-1}(\pi'(\lambda')) := \pi(\lambda' P^{-1})$ for all $\lambda' \in D^{1\times p}$, where $\pi : D^{1\times p} \longrightarrow M$ (resp., $\pi' : D^{1\times p'} \longrightarrow M'$) is the canonical projection. This result can be proved again since we have

$$\begin{pmatrix} R & -Q \\ P^{-1} & 0 \end{pmatrix} \begin{pmatrix} 0 & P \\ -Q^{-1} & R' \end{pmatrix} = I_{q+p}, \quad \begin{pmatrix} 0 & P \\ -Q^{-1} & R' \end{pmatrix} \begin{pmatrix} R & -Q \\ P^{-1} & 0 \end{pmatrix} = I_{p+q}, \tag{17}$$

and Theorem 1 then yields again the isomorphism $M \cong M'$.

Remark 3. If R is a full row rank matrix, then it is known that M is a *free* left D-module of rank $p-q$, i.e., $M \cong D^{1\times(p-q)}$, if and only if there exist $P' \in D^{(p-q)\times p}$, $P \in D^{p\times(p-q)}$, and $Z \in D^{p\times q}$ such that

$$\begin{pmatrix} R \\ P' \end{pmatrix} (Z \quad P) = I_p,$$

i.e., if and only if there exists $P' \in D^{(p-q)\times p}$ such that $(R^T \quad P'^T)^T \in \mathrm{GL}_p(D)$. For more details, see [22]. This result corresponds to the extreme case of Corollary 2 where $q' = 0$ (and thus, $p' = p-q$) and $M' = D^{1\times(p-q)}$, i.e., to the case of the following commutative exact diagram:

$$\begin{tikzcd} 0 \arrow[r] & D^{1\times q} \arrow[r, ".R"] \arrow[d] & D^{1\times p} \arrow[r, "\pi"] \arrow[d, ".P"] & M \arrow[r] \arrow[d, "f"] & 0 \\ & 0 \arrow[r] & D^{1\times(p-q)} \arrow[r, "\pi'"] & M' \arrow[r] & 0. \end{tikzcd}$$

In particular, we have $P' P = I_{p-q}$, $P P' + Z R = I_p$, and $R P = 0$. We then get $R - R Z R = (R P) P' = 0$, i.e., $(I_q - R Z) R = 0$ which yields $R Z = I_q$ since R has full row rank. Then, we have $P P' Z = Z - Z (R Z) = 0$, and thus $(P' P)(P' Z) = 0$, i.e., $P' Z = 0$, which shows again that we have the following split exact sequence (see, e.g., [22, 24]):

$$\begin{tikzcd} 0 \arrow[r] & D^{1\times q} \arrow[r, shift left, ".R"] & D^{1\times p} \arrow[l, shift left, ".Z"] \arrow[r, shift left, ".P"] & D^{1\times(p-q)} \arrow[l, shift left, ".P'"] \arrow[r] & 0. \end{tikzcd}$$

Let us consider again Theorem 1 and the following two short exact sequences

$$0 \longrightarrow \mathrm{im}_D(.R) \xrightarrow{i} D^{1\times p} \xrightarrow{\pi} M \longrightarrow 0,$$
$$0 \longrightarrow \mathrm{im}_D(.R') \xrightarrow{i'} D^{1\times p'} \xrightarrow{\pi'} M' \longrightarrow 0,$$

where i (resp., i') denotes the canonical injection into $D^{1\times p}$ (resp., $D^{1\times p'}$).

In module theory, *Schanuel's lemma* (see, e.g., [24]) asserts that $M \cong M'$ yields:

$$\mathrm{im}_D(.R) \oplus D^{1\times p'} \cong \mathrm{im}_D(.R') \oplus D^{1\times p}. \tag{18}$$

As a consequence of Theorem 1, we obtain a constructive proof of Schanuel's lemma in which the isomorphism (18) and its inverse are explicitly described.

Corollary 3. *With the notations and the assumptions of Theorem 1, if we note*

$$U := \begin{pmatrix} I_p & -P \\ P' & I_{p'} - P' P \end{pmatrix} \in \mathrm{GL}_{p+p'}(D), \quad U^{-1} = \begin{pmatrix} I_p - P P' & P \\ -P' & I_{p'} \end{pmatrix},$$

then the following homomorphism of left D-modules

$$\begin{array}{rcl} u : D^{1\times q} R \oplus D^{1\times p'} & \longrightarrow & D^{1\times p} \oplus D^{1\times q'} R' \\ (\mu R \quad \lambda') & \longmapsto & (\mu R \quad \lambda') U, \end{array} \tag{19}$$

is an isomorphism and its inverse u^{-1} is defined by:

$$\begin{array}{rcl} u^{-1} : D^{1\times p} \oplus D^{1\times q'} R' & \longrightarrow & D^{1\times q} R \oplus D^{1\times p'} \\ (\lambda \quad \mu' R') & \longmapsto & (\lambda \quad \mu' R') U^{-1}. \end{array} \tag{20}$$

Proof. Let $f \in \hom_D(M, M')$ be an isomorphism. With the notations of Theorem 1 and $P_2 := Q$, we then have the following commutative exact diagram:

$$\begin{tikzcd} D^{1\times r} \arrow[r, ".R_2"] & D^{1\times q} \arrow[r, ".R"] \arrow[d, ".P_2"] & D^{1\times p} \arrow[r, "\pi"] \arrow[d, ".P"] & M \arrow[r] \arrow[d, "f"] & 0 \\ D^{1\times r'} \arrow[r, ".R_2'"] & D^{1\times q'} \arrow[r, ".R'"] & D^{1\times p'} \arrow[r, "\pi'"] & M' \arrow[r] & 0. \end{tikzcd}$$

Using $R_2\, R = 0$, $R\, P = P_2\, R'$ yields $(R_2\, P_2)\, R' = (R_2\, R)\, P = 0$, i.e., $\mathrm{im}_D(.(R_2\, P_2)) \subseteq \ker_D(.R') = \mathrm{im}_D(.R_2')$, and thus there exists $P_3 \in D^{r \times r'}$ such that $R_2\, P_2 = P_3\, R_2'$. Similarly, with the notation $P_2' := Q'$, there exists $P_3' \in D^{r' \times r}$ such that $R_2'\, P_2' = P_3'\, R_2$ and we get the following commutative exact diagram:

$$\begin{tikzcd} D^{1\times r} \arrow[r, ".R_2"] & D^{1\times q} \arrow[r, ".R"] & D^{1\times p} \arrow[r, "\pi"] & M \arrow[r] & 0 \\ D^{1\times r'} \arrow[r, ".R_2'"] \arrow[u, ".P_3'"] & D^{1\times q'} \arrow[r, ".R'"] \arrow[u, ".P_2'"] & D^{1\times p'} \arrow[r, "\pi'"] \arrow[u, ".P'"] & M' \arrow[r] \arrow[u, "f^{-1}"] & 0. \end{tikzcd}$$

Now, if we note

$$V := \begin{pmatrix} R & -P_2 \\ P' & Z' \end{pmatrix}, \quad V' := \begin{pmatrix} Z & P \\ -P_2' & R' \end{pmatrix},$$

then we have $(R_2 \quad 0)\, V = -P_3\, (0 \quad R_2')$ and $(0 \quad R_2')\, V' = -P_3'\, (R_2 \quad 0)$. Hence, if we note

$$L := D^{1\times(q+p')}/(D^{1\times r}\, (R_2 \quad 0)), \quad L' := D^{1\times(p+q')}/(D^{1\times r'}\, (0 \quad R_2')),$$

then we have the following two commutative exact diagrams

$$\begin{tikzcd} D^{1\times r} \arrow[r, ".(R_2\ \ 0)"] \arrow[d, ".-P_3"] & D^{1\times(q+p')} \arrow[r, "\kappa"] \arrow[d, ".V"] & L \arrow[r] \arrow[d, "g"] & 0 \\ D^{1\times r'} \arrow[r, ".(0\ \ R_2')"] & D^{1\times(p+q')} \arrow[r, "\kappa'"] & L' \arrow[r] & 0, \end{tikzcd} \qquad \begin{tikzcd} D^{1\times r} \arrow[r, ".(R_2\ \ 0)"] & D^{1\times(q+p')} \arrow[r, "\kappa"] & L \arrow[r] & 0 \\ D^{1\times r'} \arrow[r, ".(0\ \ R_2')"] \arrow[u, ".-P_3'"] & D^{1\times(p+q')} \arrow[r, "\kappa'"] \arrow[u, ".V'"] & L' \arrow[r] \arrow[u, "h"] & 0, \end{tikzcd}$$

where $g \in \hom_D(L, L')$ and $h \in \hom_D(L', L)$ are respectively defined by:

$$\begin{array}{rcl} g : L & \longrightarrow & L' \\ \kappa((\mu \quad \lambda')) & \longmapsto & \kappa'((\mu\, R + \lambda'\, P' \quad -\mu\, P_2 + \lambda'\, Z')), \\ h : L' & \longrightarrow & L \\ \kappa'((\lambda \quad \mu')) & \longmapsto & \kappa((\lambda\, Z - \mu'\, P_2' \quad \lambda\, P + \mu'\, R')). \end{array}$$

Then, (14) and (15) show that $h \circ g = \mathrm{id}_L$ and $g \circ h = \mathrm{id}_{L'}$, i.e., g is an isomorphism, $h = g^{-1}$, and $L' \cong L$. Now, note that we have $\mathrm{coker}_D(.R_2) \cong \mathrm{im}_D(.R)$, $\mathrm{coker}_D(.R_2') \cong \mathrm{im}_D(.R')$, $L \cong \mathrm{im}_D(.R) \oplus D^{1\times p'}$ and $L' \cong D^{1\times p} \oplus \mathrm{im}_D(.R')$, where the last two isomorphisms are defined by:

$$\begin{array}{rcl} L & \xrightarrow{\alpha} & \mathrm{im}_D(.R) \oplus D^{1\times p'} \\ \kappa((\mu \quad \lambda')) & \longmapsto & (\mu\, R \quad \lambda'), \end{array} \qquad \begin{array}{rcl} L' & \xrightarrow{\beta} & D^{1\times p} \oplus \mathrm{im}_D(.R') \\ \kappa'((\lambda \quad \mu')) & \longmapsto & (\lambda \quad \mu'\, R'). \end{array}$$

The isomorphisms $u := \beta \circ g \circ \alpha^{-1}$ and $u^{-1} = \alpha \circ h \circ \beta^{-1}$ are then defined by:

$$\begin{array}{rcl} \mathrm{im}_D(.R) \oplus D^{1\times p'} & \xrightarrow{u} & D^{1\times p} \oplus \mathrm{im}_D(.R') \\ (\mu\, R \quad \lambda') & \longmapsto & (\mu\, R + \lambda'\, P' \quad (-\mu\, P_2 + \lambda'\, Z')\, R'), \\ D^{1\times p} \oplus \mathrm{im}_D(.R') & \xrightarrow{u^{-1}} & \mathrm{im}_D(.R) \oplus D^{1\times p'} \\ (\lambda \quad \mu'\, R') & \longmapsto & ((\lambda\, Z - \mu'\, P_2')\, R \quad \lambda\, P + \mu'\, R'). \end{array}$$

Using $P_2\, R' = R\, P$, (12), and $P_2'\, R = R'\, P'$, we obtain

$$\begin{array}{l} (-\mu\, P_2 + \lambda'\, Z')\, R' = -(\mu\, R)\, P + \lambda'\, (I_{p'} - P'\, P), \\ (\lambda\, Z - \mu'\, P_2')\, R = \lambda\, (I_p - P\, P') - (\mu'\, R')\, P', \end{array}$$

which finally yields (19) and (20). □

5 The unimodular completion problem

The *unimodular completion problem* consists in studying the possibility to inflate a matrix $R_1 \in D^{q\times p}$ into a unimodular $V \in \mathrm{GL}_{q+t}(D)$, where $q + t \geq p$. The next theorem shows that a solution to this problem induces different isomorphisms between the modules finitely presented by the matrices appearing in the inflations.

Theorem 2. *Let $p,\, q,\, s,\, t \in \mathbb{N}$ satisfy $q + t = p + s$ and $R_1 \in D^{q\times p}$, $R_2 \in D^{q\times s}$, $Q_1 \in D^{p\times t}$, $Q_2 \in D^{s\times t}$, $S_1 \in D^{p\times q}$, $S_2 \in D^{s\times q}$, $T_1 \in D^{t\times p}$, and $T_2 \in D^{t\times s}$ matrices such that:*

$$\begin{pmatrix} R_1 & R_2 \\ T_1 & T_2 \end{pmatrix} \begin{pmatrix} S_1 & Q_1 \\ S_2 & Q_2 \end{pmatrix} = I_{q+t}. \tag{21}$$

Then, we have:

$$\left\{ \begin{array}{l} \mathrm{coker}_D(.R_1) \cong \mathrm{coker}_D(.Q_2), \\ \mathrm{coker}_D(.S_1) \cong \mathrm{coker}_D(.T_2), \\ \mathrm{coker}_D(.Q_1) \cong \mathrm{coker}_D(.R_2), \\ \mathrm{coker}_D(.T_1) \cong \mathrm{coker}_D(.S_2), \end{array} \right. \qquad \left\{ \begin{array}{l} \ker_D(.R_1) \cong \ker_D(.Q_2), \\ \ker_D(.S_1) \cong \ker_D(.T_2), \\ \ker_D(.Q_1) \cong \ker_D(.R_2), \\ \ker_D(.T_1) \cong \ker_D(.S_2). \end{array} \right. \tag{22}$$

Right D-module analogs of (22) hold, i.e., we have:

$$\mathrm{coker}_D(R_1.) \cong \mathrm{coker}_D(Q_2.), \quad \ker_D(R_1.) \cong \ker_D(Q_2.), \ldots$$

Proof. By 1 of Corollary 2, the identity (21) yields the following identity:

$$\begin{pmatrix} S_1 & Q_1 \\ S_2 & Q_2 \end{pmatrix} \begin{pmatrix} R_1 & R_2 \\ T_1 & T_2 \end{pmatrix} = I_{p+s}. \tag{23}$$

From (21), we get $R_1\, Q_1 = -R_2\, Q_2$ which yields the commutative exact diagram

$$\begin{tikzcd} 0 \arrow[r] & \ker_D(.R_1) \arrow[r] \arrow[d, "\alpha_1'"] & D^{1\times q} \arrow[r, ".R_1"] \arrow[d, ".-R_2"] & D^{1\times p} \arrow[r, "\pi_1"] \arrow[d, ".Q_1"] & \mathrm{coker}_D(.R_1) \arrow[r] \arrow[d, "\alpha_1"] & 0 \\ 0 \arrow[r] & \ker_D(.Q_2) \arrow[r] & D^{1\times s} \arrow[r, ".Q_2"] & D^{1\times t} \arrow[r, "\kappa_2"] & \mathrm{coker}_D(.Q_2) \arrow[r] & 0, \end{tikzcd}$$

where α_1 and α_1' are respectively defined by:

$$\begin{array}{rcl} \alpha_1 : \mathrm{coker}_D(.R_1) & \longrightarrow & \mathrm{coker}_D(.Q_2) \\ \pi_1(\lambda_1) & \longmapsto & \kappa_2(\lambda_1\, Q_1), \end{array} \qquad \begin{array}{rcl} \alpha_1' : \ker_D(.R_1) & \longrightarrow & \ker_D(.Q_2) \\ \mu_1 & \longmapsto & -\mu_1\, R_2. \end{array} \tag{24}$$

Similarly, from (23), we get $Q_2\, T_1 = -S_2\, R_1$ which yields the following commutative exact diagram

$$\begin{tikzcd} 0 \arrow[r] & \ker_D(.Q_2) \arrow[r] \arrow[d, "\alpha_2'"] & D^{1\times s} \arrow[r, ".Q_2"] \arrow[d, ".-S_2"] & D^{1\times t} \arrow[r, "\kappa_2"] \arrow[d, ".T_1"] & \mathrm{coker}_D(.Q_2) \arrow[r] \arrow[d, "\alpha_2"] & 0 \\ 0 \arrow[r] & \ker_D(.R_1) \arrow[r] & D^{1\times q} \arrow[r, ".R_1"] & D^{1\times p} \arrow[r, "\pi_1"] & \mathrm{coker}_D(.R_1) \arrow[r] & 0, \end{tikzcd}$$

where α_2 and α_2' are respectively defined by:

$$\begin{array}{rcl} \alpha_2 : \mathrm{coker}_D(.Q_2) & \longrightarrow & \mathrm{coker}_D(.R_1) \\ \kappa_2(\nu_2) & \longmapsto & \pi_1(\nu_2\, T_1), \end{array} \qquad \begin{array}{rcl} \alpha_2' : \ker_D(.Q_2) & \longrightarrow & \ker_D(.R_1) \\ \theta_2 & \longmapsto & -\theta_2\, S_2. \end{array} \tag{25}$$

Using (21) and (23), we get $T_1\, Q_1 = I_t - T_2\, Q_2$ and $Q_1\, T_1 = I_p - S_1\, R_1$, which yields

$$\begin{cases} (\alpha_1 \circ \alpha_2)(\kappa_2(\nu_2)) = \kappa_2(\nu_2\, T_1\, Q_1) = \kappa_2(\nu_2) - \kappa_2((\nu_2\, T_2)\, Q_2) = \kappa_2(\nu_2), \\ (\alpha_2 \circ \alpha_1)(\pi_1(\lambda_1)) = \pi_1(\lambda_1\, Q_1\, T_1) = \pi_1(\lambda_1) - \pi_1((\lambda_1\, S_1)\, R_1)) = \pi_1(\lambda_1), \end{cases}$$

and shows that α_1 is an isomorphism, $\mathrm{coker}_D(.Q_2) \cong \mathrm{coker}_D(.R_1)$, and $\alpha_2 = \alpha_1^{-1}$.

Now, using (23) and (21), we get $S_2\, R_2 = I_s - Q_2\, T_2$ and $R_2\, S_2 = I_q - R_1\, S_1$, which yields

$$\begin{cases} (\alpha'_1 \circ \alpha'_2)(\theta_2) = \theta_2\, (S_2\, R_2) = \theta_2 - (\theta_2\, Q_2)\, T_2 = \theta_2, \\ (\alpha'_2 \circ \alpha'_1)(\mu_1) = \mu_1\, (R_2\, S_2) = \mu_1 - (\mu_1\, R_1)\, S_1 = \mu_1, \end{cases}$$

for all $\theta_2 \in \ker_D(.Q_2)$ and for all $\mu_1 \in \ker_D(.R_1)$, which shows that α'_1 is an isomorphism, i.e., $\ker_D(.Q_2) \cong \ker_D(.R_1)$, and $\alpha'_2 = {\alpha'_1}^{-1}$.

In the above arguments, we can exchange the role played by R_1 (resp., Q_2) by that of S_1 (resp., T_2) in the identities (21) and (23) to get the following isomorphisms

$$\begin{array}{rcl} \beta_1 : \mathrm{coker}_D(.S_1) & \longrightarrow & \mathrm{coker}_D(.T_2) \\ \sigma_1(\zeta_1) & \longmapsto & \varepsilon_2(\zeta_1\, R_2), \end{array} \qquad \begin{array}{rcl} \beta_1^{-1} : \mathrm{coker}_D(.T_2) & \longrightarrow & \mathrm{coker}_D(.S_1) \\ \varepsilon_2(\xi_2) & \longmapsto & \sigma_1(\xi_2\, S_2), \end{array} \tag{26}$$

$$\begin{array}{rcl} \beta'_1 : \ker_D(.S_1) & \longrightarrow & \ker_D(.T_2) \\ \vartheta_1 & \longmapsto & -\vartheta_1\, Q_1, \end{array} \qquad \begin{array}{rcl} {\beta'_1}^{-1} : \ker_D(.T_2) & \longrightarrow & \ker_D(.S_1) \\ \varpi_2 & \longmapsto & -\varpi_2\, T_1, \end{array} \tag{27}$$

where $\sigma_1 : D^{1\times q} \longrightarrow \mathrm{coker}_D(.S_1)$ (resp., $\varepsilon_2 : D^{1\times s} \longrightarrow \mathrm{coker}_D(.T_2)$) is the canonical projection, i.e., we have:

$$\mathrm{coker}_D(.S_1) \cong \mathrm{coker}_D(.T_2), \quad \ker_D(.S_1) \cong \ker_D(.T_2).$$

Using (23), we get $Q_1\, T_2 = -S_1\, R_2$, which yields the following commutative exact diagram

$$\begin{tikzcd} 0 \arrow[r] & \ker_D(.Q_1) \arrow[r] \arrow[d, "\gamma'_1"] & D^{1\times p} \arrow[r, ".Q_1"] \arrow[d, ".-S_1"] & D^{1\times t} \arrow[r, "\kappa_1"] \arrow[d, ".T_2"] & \mathrm{coker}_D(.Q_1) \arrow[r] \arrow[d, "\gamma_1"] & 0 \\ 0 \arrow[r] & \ker_D(.R_2) \arrow[r] & D^{1\times q} \arrow[r, ".R_2"] & D^{1\times s} \arrow[r, "\pi_2"] & \mathrm{coker}_D(.R_2) \arrow[r] & 0, \end{tikzcd}$$

where γ_1 and γ'_1 are respectively defined by:

$$\begin{array}{rcl} \gamma_1 : \mathrm{coker}_D(.Q_1) & \longrightarrow & \mathrm{coker}_D(.R_2) \\ \kappa_1(\nu_1) & \longmapsto & \pi_2(\nu_1\, T_2), \end{array} \qquad \begin{array}{rcl} \gamma'_1 : \ker_D(.Q_1) & \longrightarrow & \ker_D(.R_2) \\ \theta_1 & \longmapsto & -\theta_1\, S_1. \end{array} \tag{28}$$

Using (21), we get $R_2\, Q_2 = -R_1\, Q_1$, which yields the following commutative exact diagram

$$\begin{tikzcd} 0 \arrow[r] & \ker_D(.R_2) \arrow[r] \arrow[d, "\gamma'_2"] & D^{1\times q} \arrow[r, ".R_2"] \arrow[d, ".-R_1"] & D^{1\times s} \arrow[r, "\pi_2"] \arrow[d, ".Q_2"] & \mathrm{coker}_D(.R_2) \arrow[r] \arrow[d, "\gamma_2"] & 0 \\ 0 \arrow[r] & \ker_D(.Q_1) \arrow[r] & D^{1\times p} \arrow[r, ".Q_1"] & D^{1\times t} \arrow[r, "\kappa_1"] & \mathrm{coker}_D(.Q_1) \arrow[r] & 0, \end{tikzcd}$$

where γ_2 and γ'_2 are respectively defined by:

$$\begin{array}{rcl} \gamma_2 : \mathrm{coker}_D(.R_2) & \longrightarrow & \mathrm{coker}_D(.Q_1) \\ \pi_2(\lambda_2) & \longmapsto & \kappa_1(\lambda_2\, Q_2), \end{array} \qquad \begin{array}{rcl} \gamma'_2 : \ker_D(.R_2) & \longrightarrow & \ker_D(.Q_1) \\ \mu_2 & \longmapsto & -\mu_2\, R_1. \end{array} \tag{29}$$

Using (21) and (23), we get $T_2\, Q_2 = I_t - T_1\, Q_1$ and $Q_2\, T_2 = I_s - S_2\, R_2$, which yields

$$\begin{cases} (\gamma_2 \circ \gamma_1)(\kappa_1(\nu_1)) = \kappa_1(\nu_1\, (T_2\, Q_2)) = \kappa_1(\nu_1) - \kappa_1((\nu_1\, T_1)\, Q_1) = \kappa_1(\nu_1), \\ (\gamma_1 \circ \gamma_2)(\pi_2(\lambda_2)) = \pi_2(\lambda_2\, (Q_2\, T_2)) = \pi_2(\lambda_2) - \pi_2((\lambda_2\, S_2)\, R_2) = \pi_2(\lambda_2), \end{cases}$$

and shows that γ_1 is an isomorphism, i.e., $\operatorname{coker}_D(.Q_1) \cong \operatorname{coker}_D(.R_2)$, and $\gamma_2' = \gamma_2^{-1}$.

Using (23) and (21), we get $S_1\,R_1 = I_p - Q_1\,T_1$ and $R_1\,S_1 = I_q - R_2\,S_2$, which yields

$$\begin{cases} (\gamma_2' \circ \gamma_1')(\theta_1) = \theta_1\,(S_1\,R_1) = \theta_1 - (\theta_1\,Q_1)\,T_1 = \theta_1, \\ (\gamma_1' \circ \gamma_2')(\mu_2) = \mu_2\,(R_1\,S_1) = \mu_2 - (\mu_2\,R_2)\,S_2 = \mu_2, \end{cases}$$

for all $\theta_1 \in \ker_D(.Q_1)$ and for all $\mu_2 \in \ker_D(.R_2)$, which shows that γ_1' is an isomorphism, i.e., $\ker_D(.Q_1) \cong \ker_D(.R_2)$, and $\gamma_2' = \gamma_1'^{-1}$.

Finally, we can similarly show that we have the following isomorphisms

$$\begin{array}{rcl} \delta_1 : \operatorname{coker}_D(.T_1) & \longrightarrow & \operatorname{coker}_D(.S_2) \\ \varepsilon_1(\xi_1) & \longmapsto & \sigma_2(\xi_1\,S_1), \end{array} \qquad \begin{array}{rcl} \delta_1^{-1} : \operatorname{coker}_D(.S_2) & \longrightarrow & \operatorname{coker}_D(.T_1) \\ \sigma_2(\zeta_2) & \longmapsto & \varepsilon_1(\zeta_2\,R_1), \end{array} \tag{30}$$

$$\begin{array}{rcl} \delta_1' : \ker_D(.T_1) & \longrightarrow & \ker_D(.S_2) \\ \varpi_1 & \longmapsto & -\varpi_1\,T_2, \end{array} \qquad \begin{array}{rcl} \delta_1'^{-1} : \ker_D(.S_2) & \longrightarrow & \ker_D(.T_1) \\ \theta_2 & \longmapsto & -\theta_2\,Q_2, \end{array} \tag{31}$$

where $\varepsilon_1 : D^{1\times p} \longrightarrow \operatorname{coker}_D(.T_1)$ (resp., $\sigma_2 : D^{1\times q} \longrightarrow \operatorname{coker}_D(.S_2)$) is the canonical projection, i.e., we have:

$$\operatorname{coker}_D(.T_1) \cong \operatorname{coker}_D(.S_2), \quad \ker_D(.T_1) \cong \ker_D(.S_2).$$

Right D-module analogs of (22) can be proved similarly. □

Remark 4. When $s \leq q$ and $t = p - (q - s) > 0$, Theorem 2 shows that we have $M := \operatorname{coker}_D(.R_1) \cong \operatorname{coker}_D(.Q_2)$, where $Q_2 \in D^{s\times t}$, which yields $\ker_{\mathcal{F}}(R_1.) \cong \ker_{\mathcal{F}}(Q_2.)$ for all left D-modules $\mathcal{F}$, i.e., the linear system $R_1\,\eta = 0$ is equivalent to the linear system $Q_2\,\zeta = 0$ defined by fewer equations and fewer unknowns. Such a reduction is called *Serre's reduction* and is studied in detail in [1, 9, 11]. Theorem 2 is an extension of Theorem 4.1 of [1] for a non necessarily full row rank matrix R_1.

Example 11. With the notations $R_1 := R$, $R_2 := -Q$, $T_1 := P'$, $T_2 := Z'$, $S_1 := Z$, $S_2 := -Q'$, $Q_1 := P$, and $Q_2 := R'$, in Example 8, we proved the identity (21). By Theorem 2, we find again that $M := \operatorname{coker}_D(.R) \cong M' := \operatorname{coker}_D(.R')$, where R and R' have not full row rank (see Example 8), and $\ker_D(.R) \cong \ker_D(.R')$. We also have $\operatorname{coker}_D(.Z) \cong \operatorname{coker}_D(.Z')$ and $\ker_D(.Z) \cong \ker_D(.Z')$ (see (26) and (27)), $\operatorname{coker}_D(.Q) \cong \operatorname{coker}_D(.P)$ and $\ker_D(.Q) \cong \ker_D(.P)$ (see (28) and (29)), $\operatorname{coker}_D(.P') \cong \operatorname{coker}_D(.Q')$ and $\ker_D(.P') \cong \ker_D(.Q')$ (see (30) and (31)).

Example 12. We consider again Example 10. Theorem 2 shows that $M := \operatorname{coker}_D(.R) \cong M' := \operatorname{coker}_D(.R')$, $\operatorname{coker}_D(.P) \cong \operatorname{coker}_D(.Q) = 0$ and $\ker_D(.P) \cong \ker_D(.Q) = 0$ since $P \in \operatorname{GL}_p(D)$ and $Q \in \operatorname{GL}_q(D)$.

Example 13. We consider again Example 9, where $S_1 := Z = (0 \quad 0)$ and $T_2 := Z'$. We can check that $\operatorname{coker}_D(.Z')$ is a free left D-module of rank 5 and $\operatorname{coker}_D(.Z) \cong D^{1\times 2}$ is a free left D-module of rank 2. Hence, the isomorphisms (22) of Theorem 2 only hold when we have (21) and not when (14) and (15) hold.

We can give a system-theoretic interpretation of Theorem 2. The hypotheses of Theorem 2 show that we can inflate the linear system $R_1\,\eta_1 = 0$ into the larger linear system $R_1\,\eta_1 + R_2\,\eta_2 = 0$ which is *flat* (see [3, 13, 21] and the references therein), i.e., which is associated with the free left D-module $E := \operatorname{coker}_D(.(R_1 \quad R_2))$ of rank $t = p - q + s$ (see Remark 3). Then, we know that the flat system admits an injective parametrization, i.e., we have $\ker_{\mathcal{F}}((R_1 \quad R_2).) = \operatorname{im}_{\mathcal{F}}((Q_1^T \quad Q_2^T)^T.)$, where $T_1\,Q_1 + T_2\,Q_2 = I_t$. For more details, see [3, 22]. Hence, we get

$$R_1\,\eta_1 + R_2\,\eta_2 = 0 \quad \Leftrightarrow \quad \begin{cases} \eta_1 = Q_1\,\xi, \\ \eta_2 = Q_2\,\xi, \end{cases}$$

for a certain $\xi \in \mathcal{F}^t$ which is such that $\xi = T_1\,\eta_1 + T_2\,\eta_2$. Now, setting $\eta_2 = 0$, we get that for $\eta_1 \in \ker_{\mathcal{F}}(R_1.)$, there exists a unique $\xi = T_1\,\eta_1 \in \mathcal{F}^t$ such that:

$$\eta_1 = Q_1\,\xi, \quad Q_2\,\xi = 0.$$

Within systems theory, we find again the first isomorphisms of (24) and (25).

For instance, the linear OD system $\dot{x}(t) = A\,x(t)$, with $A \in \mathbb{R}^{n\times n}$, is equivalent to an ODE with constant coefficients in one unknown if and only if there exists $B \in \mathbb{R}^n$ such that the control (inflated) linear system $\dot{x}(t) = A\,x(t) + B\,u(t)$ is flat, i.e., if and only if it is controllable. For more details and extensions, see [9].

Remark 5. We can give another (pictorial) proof of the first point of Theorem 2, i.e., of $\operatorname{coker}_D(.R_1) \cong \operatorname{coker}_D(.Q_2)$ and $\ker_D(.R_1) \cong \ker_D(.Q_2)$. Identities (21) and (23) are equivalent to the following split short exact sequence of left D-modules:

$$\begin{tikzcd} 0 \arrow[r] & D^{1\times q} \arrow[r, shift left, ".(R_1 \quad R_2)"] & D^{1\times (p+s)} \arrow[l, shift left, ".\left(\begin{array}{c} S_1 \\ S_2 \end{array}\right)"] \arrow[r, shift left, ".\left(\begin{array}{c} Q_1 \\ Q_2 \end{array}\right)"] & D^{1\times t} \arrow[l, shift left, ".(T_1 \quad T_2)"] \arrow[r] & 0. \end{tikzcd} \tag{32}$$

For more details, see, e.g., [21, 24]. With the above notations, we then have the following commutative exact diagram:

$$\begin{tikzcd}
 & & & & 0 \arrow[d] & \\
 & & & 0 \arrow[d] & \ker_D(.R_1) \arrow[d] & \\
 & & 0 \arrow[r] \arrow[d] & D^{1\times q} \arrow[r, equal] \arrow[d, shift left, ".(R_1 \quad R_2)"] & D^{1\times q} \arrow[r] \arrow[d, ".R_1"] & 0 \\
 & 0 \arrow[r] & D^{1\times s} \arrow[r, shift left, ".(0 \quad I_s)"] \arrow[d, equal] & D^{1\times (p+s)} \arrow[l, shift left, ".(0^T \quad I_s^T)^T"] \arrow[u, shift left, ".\left(\begin{array}{c} S_1 \\ S_2 \end{array}\right)"] \arrow[r, shift left, ".(I_p^T \quad 0^T)^T"] \arrow[d, shift left, ".\left(\begin{array}{c} Q_1 \\ Q_2 \end{array}\right)"] & D^{1\times p} \arrow[l, shift left, ".(I_p \quad 0)"] \arrow[r] \arrow[d, "\pi_1"] & 0 \\
0 \arrow[r] & \ker_D(.Q_2) \arrow[r] & D^{1\times s} \arrow[r, ".Q_2"] \arrow[d] & D^{1\times t} \arrow[r, "\beta_1"] \arrow[u, shift left, ".(T_1 \quad T_2)"] \arrow[d] & M \arrow[r] \arrow[d] & 0. \\
 & & 0 & 0 & 0 &
\end{tikzcd} \tag{33}$$

Let us denote

$$L := \operatorname{coker}_D(.Q_2) = D^{1\times t}/(D^{1\times s}\, Q_2), \; M := \operatorname{coker}_D(.R_1) = D^{1\times p}/(D^{1\times q}\, R_1),$$

and $\kappa_2 : D^{1\times t} \longrightarrow L$ (resp., $\pi_1 : D^{1\times p} \longrightarrow M$) the canonical projection. Then, using (33), we obtain the following isomorphism:

$$\begin{array}{rcl} \phi : L & \longrightarrow & M \\ \kappa_2(\nu_2) & \longmapsto & \pi_1\left(\nu_2\,(T_1 \quad T_2)\left(\begin{array}{c} I_p \\ 0 \end{array}\right)\right) = \pi_1(\nu_2\, T_1), \\ \phi^{-1} : M & \longrightarrow & L \\ \pi_1(\lambda_1) & \longmapsto & \kappa_2\left(\lambda_1\,(I_p \quad 0)\left(\begin{array}{c} Q_1 \\ Q_2 \end{array}\right)\right) = \kappa_2(\lambda_1\, Q_1). \end{array}$$

A *chase* in the commutative exact diagram (33) (see, e.g., [24]) yields the following isomorphism

$$\begin{array}{rclcrcl} \gamma : \ker_D(.Q_2) & \longrightarrow & \ker_D(.R_1) & \qquad & \gamma^{-1} : \ker_D(.R_1) & \longrightarrow & \ker_D(.Q_2) \\ \theta_2 & \longmapsto & \theta_2\, S_2, & & \mu_1 & \longmapsto & \mu_1\, R_2, \end{array}$$

i.e., we have $\ker_D(.R_1) \cong \ker_D(.Q_2)$. Finally, the other isomorphisms (22) stated in Theorem 2 can be proved similarly.

Corollary 4. *The following two assertions are equivalent:*

1. *The matrices $R \in D^{q\times p}$ and $R' \in D^{q\times p}$ are equivalent, namely, there exist $P \in \mathrm{GL}_p(D)$ and $Q \in \mathrm{GL}_q(D)$ such that $R' = Q^{-1}\, R\, P$.*

2. *There exist $Q \in \mathrm{GL}_q(D)$ and $U \in \mathrm{GL}_{p+q}(D)$ having R' as lower $q \times p$ corner such that:*

$$(R \quad -Q)\, U = (I_q \quad 0).$$

Proof. $1 \Rightarrow 2$. If R and R' are equivalent, then 2 is proved in Example 10 (see (17)).

$2 \Rightarrow 1$. Let us note:

$$U := \left(\begin{array}{cc} Z & P \\ -Q' & R' \end{array}\right), \quad U^{-1} := \left(\begin{array}{cc} R & -Q \\ P' & Z' \end{array}\right).$$

In particular, we have $R\,P = Q\,R'$, i.e., $R' = Q^{-1}\,R\,P$ since $Q \in \mathrm{GL}_q(D)$. Now, (22) yields $\mathrm{coker}_D(.P) \cong \mathrm{coker}_D(.Q) = 0$ and $\ker_D(.P) \cong \ker_D(.Q) = 0$ since $Q \in \mathrm{GL}_q(D)$, which shows that $P \in \mathrm{GL}_p(D)$ and proves 1. Finally, using $P'\,P + Z'\,R' = I_p$, we get $(P' + Z'\,Q^{-1}\,R)\,P = I_p$ which shows that we have:

$$P^{-1} = P' + Z'\,Q^{-1}\,R.$$

□

Finally, let us give an application of Theorem 2 for the study of *doubly coprime factorizations* (see, e.g., [25]). To keep the standard notations used within the *fractional representational approach* [25], we now denote the ring D by A.

Corollary 5. *Let A be an integral domain, namely, a commutative ring with no non-zero divisors,*

$$K := \left\{\frac{n}{d} \mid 0 \neq d,\, n \in A\right\}$$

the quotient field of A, $P \in K^{q\times r}$, and $P = D^{-1}\,N = \widetilde{N}\,\widetilde{D}^{-1}$ a doubly coprime factorization of P, namely, $D \in A^{q\times q}$, $N \in A^{q\times r}$, $\widetilde{D} \in A^{r\times r}$, and $\widetilde{N} \in A^{q\times r}$ satisfying the following identity

$$\left(\begin{array}{cc} D & -N \\ -\widetilde{Y} & \widetilde{X} \end{array}\right)\left(\begin{array}{cc} X & \widetilde{N} \\ Y & \widetilde{D} \end{array}\right) = I_{q+r},$$

for some matrices $X \in A^{q\times q}$, $Y \in A^{r\times q}$, $\widetilde{X} \in A^{r\times r}$, and $\widetilde{Y} \in A^{r\times q}$. Then, we have the following isomorphisms of A-modules:

$$\left\{\begin{array}{l} \mathrm{coker}_A(.D) \cong \mathrm{coker}_A(.\widetilde{D}), \\ \mathrm{coker}_A(.X) \cong \mathrm{coker}_A(.\widetilde{X}), \\ \mathrm{coker}_A(.N) \cong \mathrm{coker}_A(.\widetilde{N}), \\ \mathrm{coker}_A(.Y) \cong \mathrm{coker}_A(.\widetilde{Y}), \end{array}\right. , \quad \left\{\begin{array}{l} \ker_A(.D) \cong \ker_A(.\widetilde{D}) = 0, \\ \ker_A(.X) \cong \ker_A(.\widetilde{X}), \\ \ker_A(.N) \cong \ker_A(.\widetilde{N}), \\ \ker_A(.Y) \cong \ker_A(.\widetilde{Y}). \end{array}\right.$$

Similar results hold for right matrix multiplication, i.e., we also have:

$$\mathrm{coker}_A(D.) \cong \mathrm{coker}_A(\widetilde{D}.),\ \ker_A(D.) \cong \ker_A(\widetilde{D}.) = 0,\ \ldots$$

References

[1] M. S. Boudellioua, A. Quadrat. *Serre's reduction of linear functional systems.* Math. Comput. Sci., 2-3, 289-312, 2010.

[2] W. Bryc, G. Letac. *Meixner matrix ensembles.* J. Theor. Probab., 26, 107-152, 2013.

[3] F. Chyzak, A. Quadrat, and D. Robertz. *Effective algorithms for parametrizing linear control systems over Ore algebras.* Appl. Algebra Engrg. Comm. Comput., 16, 319-376, 2005.

[4] F. Chyzak, A. Quadrat, and D. Robertz. OREMODULES*: A symbolic package for the study of multidimensional linear systems.* Springer, Lecture Notes in Control and Inform. Sci., 352, 233-264, 2007. `http://wwwb.math.rwth-aachen.de/OreModules`.

[5] F. Chyzak, B. Salvy. *Non-commutative elimination in Ore algebras proves multivariate identities.* J. Symbolic Comput., 26, 187-227, 1998.

[6] T. Cluzeau, A. Quadrat. *Factoring and decomposing a class of linear functional systems.* Linear Algebra Appl., 428, 324-381, 2008.

[7] T. Cluzeau, A. Quadrat. OREMORPHISMS*: A homological algebraic package for factoring, reducing and decomposing linear functional systems.* Springer, Lecture Notes in Control and Inform. Sci., 388, 179-194, 2009.
`http://pages.saclay.inria.fr/alban.quadrat/OreMorphisms.html`.

[8] T. Cluzeau, A. Quadrat. A constructive version of Fitting's theorem on isomorphisms and equivalences of linear systems. Proceedings of nDS'11, Poitiers (France) (05-07/09/11).

[9] T. Cluzeau, A. Quadrat. Serre's reduction of linear systems of partial differential equations with holonomic adjoints. J. Symbolic Comput., 47, 1192-1213, 2012.

[10] T. Cluzeau, A. Quadrat. Isomorphisms and Serre's reduction of linear systems. Proceedings of nDS13, Erlangen (Germany) (9-11/09/13).

[11] T. Cluzeau, A. Quadrat. A new insight into Serre's reduction problem. Linear Algebra Appl., 483, 40-100, 2015.

[12] T. Cluzeau, A. Quadrat, M. Tõnso. OREALGEBRAICANALYSIS*: A Mathematica package for the algorithmic study of linear functional systems.* OREALGEBRAICANALYSIS project, 2015. `http://pages.saclay.inria.fr/alban.quadrat/OreAlgebraicAnalysis.html`.

[13] M. Fliess, H. Mounier. Controllability and observability of linear delay systems: an algebraic approach. ESAIM COCV 3, 301-314,1998.

[14] R. Hotta, K. Takeuchi, T. Tanisaki. *D-Modules, Perverse Sheaves, and Representation Theory.* Progress in Mathematics 236, Birkhäuser, 2008.

[15] M. Kashiwara. *Algebraic Study of Systems of Partial Differential Equations.* Mémoires de la Société Mathématique de France 63, English translation (Kyoto 1970), 1995.

[16] T. Y. Lam. *Lectures on modules and rings.* Graduate Texts in Mathematics, 189, Springer, 1999.

[17] B. Malgrange. *Systèmes différentiels à coefficients constants.* Séminaire Bourbaki 1962/63, 1-11, 1962.

[18] J. C. McConnell, J. C. Robson, *Noncommutative Noetherian Rings*, American Mathematical Society, 2000.

[19] U. Oberst. *Multidimensional constant linear systems.* Acta Appl. Math., 20, 1-175, 1990.

[20] D. Pal, H. K. Pillai. Algorithms for the theory of restrictions of scalar nD systems to proper subspaces of $\mathbb{R}^n$. Multidim. Syst. Sign. Process. 26, 439-457, 2015.

[21] A. Quadrat. *An Introduction to Constructive Algebraic Analysis and its Applications.* Les cours du CIRM, Journées Nationales de Calcul Formel (2010), 1(2), 281-471, 2010, `http://ccirm.cedram.org/ccirm-bin/fitem?id=CCIRM_2010__1_2_281_0`.

[22] A. Quadrat, D. Robertz. *Computation of bases of free modules over the Weyl algebras.* J. Symbolic Comput., 42, 1113-1141, 2007.

[23] D. Robertz. *Recent progress in an algebraic analysis approach to linear systems.* Multidimensional Systems and Signal Processing 26, 349–388, 2015.

[24] J. J. Rotman. *An Introduction to Homological Algebra.* Academic Press, 1979.

[25] M. Vidyasagar. *Control System Synthesis: A Factorization Approach*. MIT Press, 1985.

RESEARCH CENTRE
LILLE – NORD EUROPE

Parc scientifique de la Haute-Borne
40 avenue Halley - Bât A - Park Plaza
59650 Villeneuve d'Ascq

Publisher
Inria
Domaine de Voluceau - Rocquencourt
BP 105 - 78153 Le Chesnay Cedex
inria.fr

ISSN 0249-6399